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The Gift of

WILLIAM H. BUTTS, Ph.D.

A.B. 1878 A.M. 1879

Teacher of Mathematics

1898 to 1922

Assistant Dean, College of Engineering

1908 to 1922

Professor Emeritus

1922

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Observatory, Oxford.

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Emerson, William
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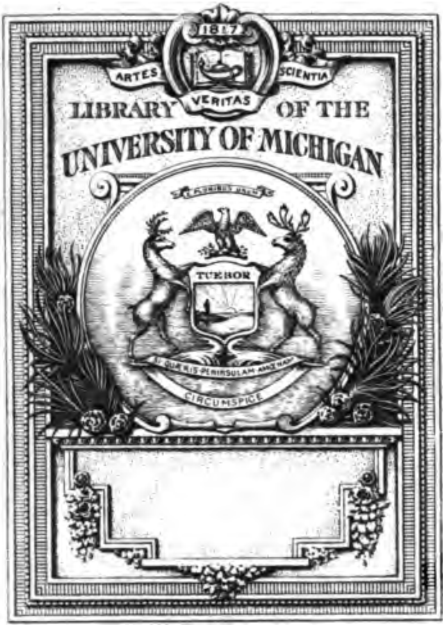

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THE
DOCTRINE
OF
PROPORTION,
ARITHMETICAL
AND
GEOMETRICAL.

Together with a general Method of arguing
by proportional Quantities.

Si Proportionis Doctrinam e Matheſi abſtuleris, nihil
fere præclarum aut egregium relinques.
Wh. Tac. Eucl.

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THE

PREFACE.

2-27-39 HCM

SINCE all manner of quantities require to be compared together, in mathematical computations, and their various relations ſearched out and determined; and as moſt of our knowledge in mathematical ſubjects depends on the proportions of ſeveral things to one another: ſo it is requiſite that the nature of proportion, and the methods of reaſoning thereby, be diſtinctly laid down and well underſtood. It is a method of reaſoning ſo very ſhort, ſubtle, ſolid, and certain, and likewiſe ſo uſeful in all parts of the mathematics, that it is impoſſible to make the leaſt progreſs without it. It is the marrow of the mathematics, and the very ſoul of geometry and geometrical reaſoning. Therefore it is abſolutely neceſſary, that every one who expects to ſucceed in his mathematical ſtudies, ſhould make himſelf acquainted with the nature of reaſoning with proportional quantities, and become ready and quick in the uſe thereof.

I had before, in the Treatiſe of Arithmetic, demonſtrated ſome few things relating to proportions; but no more than I had then preſent occaſion for, in treating of the properties of numbers. But in this ſmall tract, I have demonſtrated the doctrine of proportions univerſally, for all quantities whatſoever, as well as numbers.

The method I have here followed is this: Sect. I. treats of arithmetical proportion and progreſſion. And Sect. II. of geometrical proportion. And herein I have taken the liberty to deviate from Euclid, *by giving a different definition of proportional quantities; his being abſtruſe and unintelligible, eſpecially to young ſtudents. This here laid down being evidently agreeable to, and deducible from, the firſt, ſimple, and natural idea we form of proportion. Neither have I followed his order of propoſitions, or method of demonſtration: but have omitted many of his propoſitions as of little uſe, and added ſeveral other more uſeful ones, which he had not. And theſe I have demonſtrated from that moſt ſimple idea of proportion before mentioned, with the greateſt eaſe and perſpicuity imaginable. And becauſe the method of arguing by a general proportion is vaſtly ſhorter and eaſier than the common way with four terms; therefore I have in Sect. III. demonſtrated the fundamental propoſitions it depends on; and has ſhewn and explained the way of proceeding, according to that method. And therefore I hope this will both inſtruct and delight the diligent reader.*

W. Emerſon.

AXIOMS.

1. The whole is equal to all the parts taken together.

2. If equal quantities be added to equal quantities; the ſums will be equal.

3. If equal quantities be taken from equal quantities; the remainders will be equal.

4. If equal quantities be equally multiplied; the products will be equal.

5. If equal quantities be divided by equal numbers; the quotients will be equal.

6. Equal quantities have the ſame proportion to any third quantity: and any quantity has the ſame ratio to equal quantities.

7. Thoſe quantities are equal, that have the ſame ratio to any third; or when a third has the ſame ratio to each of them.

8. Thoſe ratios or quantities, that are equal to a third, are equal to one another.

9. A greater quantity has a greater ratio to a third, than a leſſer quantity has. And that which has the greater ratio, is the greater quantity.

10. If there be two equal ratios, and one be greater than a third, the other will be greater; if leſs, the other will be leſs.

The ſignification of the Signs or Characters here uſed.

$+$	to be added.
$-$	to be ſubtracted.
$\times$	multiplied by, or AB is A multiplied by B.
$\div$	divided by, or $\frac{A}{B}$ is A divided by B.
$=$	equal to.
$::$	geometrical proportion, as A : B : : C : D, ſignifies A is to B, as C is to D.
$\propto$	is as; a mark of general proportion.
∺	continual proportion, or geometrical progreſſion. As A : B : C : D ∺, ſignifies that A is to B, as B to C, as C to D, *&c.*
∵	arithmetical proportion, as A . B ∵ C . D.
$\div$	arithmetical progreſſion.
∴	harmonic proportion.
∻	harmonic progreſſion.

SECT. I.

Arithmetical Proportion.

DEFINITIONS.

1. *ARITHMETIC proportion*, is the relation that two quantities, of the ſame kind, have to one another, in reſpect of their difference. The former quantity is called the *antecedent*; and the latter, the *conſequent*. And theſe are called the *terms* of the proportion.

2. *Ratio* is the difference between the antecedent and conſequent. Therefore arithmetic ratio is of the ſame kind as the quantities themſelves. This is commonly called the *common difference*.

3. *Quantities arithmetically proportional*, are thoſe that have the ſame arithmetic ratio, when compared two and two; ſo that the antecedents, may be every where ſubtracted from the conſequents; or elſe the conſequents from the antecedents.

4. Continued proportion is when the firſt has the ſame proportion to the ſecond, as the ſecond to the third.

5. *Arithmetical progreſſion*, is when a ſeries of quantities are in the ſame arithmetical proportion. Or when they increaſe, or decreaſe by equal differences.

6. *Muſical proportion, and progreſſion*, is when there is a ſeries of quantities, where the numerators are the ſame, and the denominators in arithmetic progreſſion.

PROP. I.

If four quantities are arithmetically proportional, $A . B \therefore C . D$; *the ſum of the extremes is equal to the ſum of the means,* $A + D = B + C$.

For $A - B = C - D$ (Def. 3), and adding $B + D$, $A - B + B + D = C + B - D + D$ (Ax. 2); that is, $A + D = C + B$.

Cor. *If three quantities be in arithmetic progreſſion, the ſum of the extremes is double the mean.*

PROP. II.

If there be two ranks of quantities in arithmetic proportion; their ſums and differences ſhall alſo be in arithmetic proportion. If $A . B \therefore C . D$, *and* $P . Q \therefore R . S$; *then* $A + P . B + Q \therefore C + R . D + S$, *and* $A - P . B - Q \therefore C - R . D - S$.

For let $A - B = C - D = m$, and $Q - P = S - R = n$. Then $B = A - m$, $D = C - m$, $Q = P + n$, $S = R + n$. And $B + Q = A - m + P + n$, $D + S = C - m + R + n$. But $A + P . A - m + P - n \therefore C + R . C - m + R + n$ (Def. 3).

Again, $B - Q = A - m - P - n$, and $D - S = C - m - R - n$. But $A - P . A - m - P - n \therefore C - R . C - m - R - n$ (Def. 3).

PROP. III.

If three quantities are in arithmetic progreſſion; the rectangle of the extremes, together with the ſquare of the common difference, is equal to the ſquare of the middle term. If $A . B . C \div$, *then* $AC + \overline{B - A}^2 = BB$.

For let $D = B - A = C - B$, and $A = B - D$, $C = B + D$; then $AC = \overline{B - D} \times \overline{B + D} = BB + BD - BD - DD = BB - DD$. And $AC - DD = BB$ (Ax. 2).

Cor. *A ſet of arithmetical proportionals, whoſe common difference is exceeding ſmall, is nearly a ſet of geometrical proportionals.* See the next ſection.

PROP. IV.

In a ſeries of quantities in arithmetical progreſſion; the ſum of the extremes is equal to the ſum of any two means, equally diſtant from the extremes. If $A . B . C . D . E . F . G \div$, *then* $A + G = B + F = C + E$, &c.

For ſince $A . B \because F . G$ (Def. 5), therefore $A + G = B + F$ (Prop. 1). And ſince $B . C \because E . F$, therefore $C + E = B + F = A + G$, &c.

Cor. *Hence the ſum of the extremes is double the mean, when the number of terms is odd.*

PROP. V.

If out of a ſeries of quantities in arithmetical progreſſion, there be taken any ſeries of equidiſtant terms; this ſeries will alſo be in arithmetic progreſſion.

If $A . B . C . D . E . F . G . H . I . K . L . M \div$,
then $B \quad . \quad E \quad . \quad H \quad . \quad L$ *are* $\div$.

For $C - B = D - C = E - D = R$, and adding all together, $E - B = 3R$.
Alſo $F - E = G - F = H - G = R$, and $H - E = 3R$.
Again, $I - H = K - I = L - K = R$, and $L - H = 3R$, &c.
Therefore, $E - B = H - E = L - H$ (Ax. 8).
And $B . E . H . L \div$ (Def. 5).

PROP.

from the third to the fourth, as if A, B, C, D, be four quantities; then it is *directly* A : B :: C : D.

7. *Reciprocal or inverſe proportion*, is when one ſort of quantity increaſes, in the ſame proportion that another decreaſes.

8. *Diſcreet proportion*, is when out of four terms, the ſecond has not the ſame proportion to the third, which the firſt has to the ſecond, or the third to the fourth.

9. *Continual proportion*, is when the firſt term has the ſame proportion to the ſecond, as the ſecond to the third.

10. *Geometrical progreſſion*, is when a ſet of quantities are in continual proportion; or when the firſt has the ſame ratio to the ſecond, as the ſecond to the third, and as the third to the fourth, and the fourth to the fifth, *&c.*

11. *Extreme and mean ratio*, is when a quantity is ſo divided, that the leſſer part, the greater part, and the whole, are in continual proportion.

12. *Complicate ratio*, is that which ariſes by multiplying ſeveral other ratios together.

13. *Duplicate*, *triplicate*, *ratio*, &c. is the ſquare, cube, *&c.* of ſome given ratio.

14. *Harmonical ratio*, is when a quantity is divided into three parts, ſo that the whole is to one part, as the ſecond part to the third. And when the ſecond and third are equal; it is called *harmonic proportion continued.*

PROP. I.

If ſeveral pairs of quantities are in the ſame proportion, A : B : : C : D : : E : F : : G : H; *then as any antecedent to its conſequent, ſo is any other antecedent to its conſequent,* A : B : : G : H.

For ſince $\frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{G}{H}$ (Def. 4), therefore $\frac{A}{B} = \frac{G}{H}$ (Ax. 8); whence A : B : : G : H (Def. 4).

PROP. II.

If four quantities are proportional, A : B : : C : D; *and if the firſt* A, *be greater than the ſecond* B; *then the third* C, *ſhall be greater than the fourth* D. *If equal, they ſhall be equal; if leſs, leſs.*

For ſince $\frac{A}{B} = \frac{C}{D}$ (Def. 4), by the nature of fractional quantities, if A be greater than B, the quotient or ratio will be more than 1, and therefore C greater than D. But if A be equal to B, $\frac{A}{B} = 1$, and C = D. But if A be leſs than B, the quotient is leſs than 1, and therefore C leſs than D.

PROP. III.

If four quantities are proportional, A : B : : C : D; *they ſhall alſo be proportional by reverſion; that is, the ſecond* B *is to the firſt* A; *as the fourth* D, *is to the third* C; *or* B : A : : D : C.

For let $\frac{A}{B} = \frac{C}{D} = r$ the ratio, then $A = Br$, and $C = Dr$ (Ax. 4); and $B = \frac{A}{r}$, and $D = \frac{C}{r}$ (Ax. 5); alſo $\frac{B}{A} = \frac{1}{r}$, and $\frac{D}{C} = \frac{1}{r}$ (ib.); whence $\frac{B}{A} = \frac{D}{C}$ (Ax. 8); therefore B : A : : D : C (Def. 4).

PROP. IV.

If four quantities of the ſame kind are proportional, A : B :: C : D; *they ſhall be proportional alternately or by permutation; that is, the firſt* A, *ſhall be to the third* C; *as the ſecond* B, *is to the fourth* D.

For let $\frac{A}{B} = \frac{C}{D} = r$, then $A = Br$, and $C = Dr$ (Ax. 4); then $\frac{A}{C} = \frac{Br}{Dr} = \frac{B}{D}$ (Ax. 5); therefore A : C :: B : D (Def. 4).

PROP. V.

Quantities are in the ſame ratio, as their equimultiples; A : B :: nA : nB.

For let $\frac{A}{B} = r$, then $A = Br$ (Ax. 4); and $nA = nBr$ (ib.); and $\frac{nA}{nB} = r$ (Ax. 5); therefore $\frac{A}{B} = \frac{nA}{nB}$ (Ax. 8); therefore A : B :: nA : nB.

Cor. 1. *Quantities are in the ſame ratio, as their like parts.*

For $nA : nB :: \frac{nA}{n} : \frac{nB}{n} :: A : B$.

Cor. 2. *The like parts of two quantities, taken an equal number of times, are as the quantities themſelves.*

PROP. VI.

If four quantities are proportional, A : B :: C : D; *and two homologous or analogous terms be both of them equally multiplied, or divided; the four terms will ſtill be proportional.*

For $C : D :: nC : nD$ (Pr. V) $:: \frac{C}{n} : \frac{D}{n}$ (Pr. V. Cor. 1); therefore $A : B :: nC : nD :: \frac{C}{n} : \frac{D}{n}$ (Prop. I).

Again, $A : C :: B : D$ (Prop. IV) $:: nB : nD :: \frac{B}{n} : \frac{D}{n}$ (Prop. VI). Therefore $A : nB :: C : nD$. And $A : \frac{B}{n} :: C : \frac{D}{n}$ (Prop. IV).

Cor. 1. *If two correſpondent terms be multiplied by one number, and the other two terms by another number; the reſulting terms will be proportional: If* $A : B :: C : D$, *then* $mA : mB :: nC : nD$; *or* $mA : nB :: mC : nD$.

Cor. 2. *And if two correſpondent terms be divided by one number, and the other two by another number; the quotients will be proportional.*

Cor. 3. *Hence, inſtead of any two correſpondent terms; two others, proportional to them, may be put in their room.*

PROP. VII.

If four quantities are proportional; and inſtead of two factors, in two analogous terms, if there be ſubſtituted two other quantities, in the ſame ratio; the four quantities will ſtill be proportional: If $A : B :: PQ : RS$; *and* $Q : S :: M : N$. *Then* $A : B :: PM : RN$.

For ſince $A : B :: PQ : RS$; by dividing the antecedents by P, and the conſequents by R, $\frac{A}{P} : \frac{B}{R} :: Q : S :: M : N$ (Prop. VI. Cor.); then multiplying the antecedents by P, and the conſequents by R, we have $A : B :: PM : RN$.

PROP. VIII.

If the parts taken away from two whole quantities, be as the wholes; then the remainders, ſhall be as the wholes. If $A : C :: A + B : C + D$; *then* $B : D :: A + B : C + D$.

For $A : A + B :: C : C + D$ (Prop. IV); and $A + B : A :: C + D : C$ (Prop. III); and $\frac{A + B}{A} = \frac{C + D}{C}$ (Def. 4); that is, $1 + \frac{B}{A} = 1 + \frac{D}{C}$, and $\frac{B}{A} = \frac{D}{C}$ (Ax. 3); therefore $B : A :: D : C$, and $B : D :: A : C$ (Prop. IV) $:: A + B : C + D$.

Cor. *The ſame things ſuppoſed, the remainders ſhall be as the parts taken away,* $A : B :: C : D$.

PROP. IX.

The ſum of the greateſt and leaſt, of four proportional quantities, is greater than the ſum of the other two.

Suppoſe $A : B :: C : D$, and let A be the greateſt term, then of conſequence D is the leaſt (Prop. II): then $\frac{A}{B} = \frac{C}{D} = r$. Now ſince A is greater than B, r is greater than 1, therefore put $r = 1 + s$. Whence $A = rB = B + sB$, and $C = rD = D + sD$ (Ax. 8). Then $A + D = B + sB + D$, and $B + C = B + D + sD$. But B is greater than D, and sB greater than sD; therefore $B + D + sB$ is greater than $B + D + sD$; or $A + D$ greater than $B + C$.

Cor. *The ſum of* A *and* D = *ſum of* B *and* C $+ \overline{r - 1} \times \overline{B - D}$.

For one of theſe ſums exceeds the other by $s \times \overline{B - D}$.

PROP. X.

If ſeveral quantities are proportional; A : B :: C : D :: E : F :: G : H; *as one of the antecedents, to its conſequent*; *ſo is the ſum of all the antecedents, to the ſum of all the conſequents*; A : B : A + C + E + G : B + D + F + H.

For let $\frac{A}{B} = r$, or $A = Br$, $C = Dr$, $E = Fr$, $G = Hr$, and $A + C + E + G = Br + Dr + Fr + Hr = \overline{B + D + F + H} \times r$ (Ax. 8); therefore $\frac{A + C + E + G}{B + D + F + H} = \frac{\overline{B + D + F + H} \times r}{B + D + F + H} = r$; therefore $\frac{A}{B} = \frac{A + C + E + G}{B + D + F + H}$; therefore, &c.

PROP. XI.

If there be two ranks of proportional quantities, and the two means be the ſame in both; *the extremes will be reciprocally proportional. If* A : B :: C : D, *and* E : B :: C : F; *then* A : E :: F : D.

For let $\frac{A}{B} = \frac{C}{D} = r$, and ſince B : E :: F : C (Pr. III); therefore let $\frac{B}{E} = \frac{F}{C} = s$. Then $rs = \frac{A}{B} \times \frac{B}{E} = \frac{C}{D} \times \frac{F}{C}$ (Ax. 4); that is, $\frac{A}{E} = \frac{F}{D}$; or A : E :: F : D.

Cor. *In two ranks of proportional quantities, if the extremes be the ſame in both*; *the means will be reciprocally proportional.*

For if B : A :: D : C, and B : E :: F : C; then by reverſion A : B :: C : D, and E : B :: C : F. Whence A : E :: F : D (Prop. XI).

PROP. XII.

If four quantities are proportional; A : B :: C : D; *the product of the extremes is equal to the product of the means,* AD = BC.

For let $\frac{A}{B} = \frac{C}{D} = r$, then A = B$r$, and C = D$r$ (Ax. 4); whence AD = BrD, and BC = BrD (Ax. 4); therefore AD = BC (Ax. 8).

Cor. 1. *If two products are equal,* AD = BC; *the ſides or factors are reciprocally proportional,* A : B :: C : D.

For let A : B : C : Q, then AQ = BC (Prop. XII) = AD (hyp.); therefore Q = D (Ax. 5); and A : B :: C : D (Ax. 7).

Cor. 2. *If three quantities are continually proportional; the rectangle of the extremes is equal to the ſquare of the mean. And the contrary.*

Cor. 3. *In four proportional quantities, if one extreme be multiplied by any number, and the other extreme, divided by it; the quantities will ſtill be proportional. The ſame holds of the means. Conſequently any two factors in the two extremes may change places; or in the two means.*

For if A : B :: C : D, then AD = BC, and nAD = nBC (Ax. 4); then nA : B :: nC : D (Cor. 1) :: C : $\frac{D}{n}$ (Cor. 1. Prop. 5).

SCHOLIUM.

It is ſuppoſed here that two analogous terms are numbers, or at leaſt, that they are repreſented by numbers.

PROP. XIII.

If four quantities are proportional, $A : B :: C : D$; *and if the analogous terms be compounded any way by addition or ſubtraction; ſo that both pairs be ordered alike; then they will ſtill be proportional.*

If $A : B :: C : D$.

Then $A : A + B :: C : C + D$.

$A : A - B :: C : C - D$.

$A : B - A :: C : D - C$.

$A + B : B :: C + D : D$.

$A - B : B :: C - D : D$.

$B - A : B :: D - C : D$.

$A + B : A - B :: C + D : C - D$.

$A + B : B - A :: C + D : D - C$.

$A : B :: A + C : B + D$.

$A : B :: A - C : B - D$, &c. and the reverſe thereof.

For in any caſe, the product of the means is equal to the product of the extremes.

Cor. *When the quantities are compounded after any of the foregoing ways, then it will be,* $A : B :: C : D$.

PROP. XIV.

If one quantity has the ſame proportion to ſeveral quantities ſeparately; as a ſecond quantity has to as many others: then the firſt has the ſame proportion to the ſum of the firſt ſet, as the ſecond has to the ſum of the laſt ſet.

If $A : \begin{cases} B \\ C \\ D \end{cases} :: F : \begin{cases} G \\ H \\ I \end{cases}$ *then* $A : B + C + D :: F : G + H + I$.

For $\left.\begin{matrix}B\\C\\D\end{matrix}\right\} : A :: \left.\begin{matrix}G\\H\\I\end{matrix}\right\} : F$ (Prop. III), then $\frac{B}{A} = \frac{G}{F}$, $\frac{C}{A} = \frac{H}{F}$, $\frac{D}{A} = \frac{I}{F}$ (Def. 4). Therefore $\frac{B}{A} + \frac{C}{A} + \frac{D}{A}$ or $\frac{B+C+D}{A} = \frac{G}{F} + \frac{H}{F} + \frac{I}{F}$ or $\frac{G+H+I}{F}$ (Ax. 2); therefore $B+C+D : A :: G+H+I : F$ (Def. 4); and $A : B+C+D :: F : G+H+I$ (Prop. III).

Cor. 1. *If one quantity be ſeparately to two quantities; as a ſecond is to two others: the firſt will be to the difference of the firſt two; as the ſecond, is to the difference of the laſt two.*

If $A : \left\{\begin{matrix}B\\C\end{matrix}\right. :: F : \left\{\begin{matrix}G\\H\end{matrix}\right.$. *Then* $A : B-C :: F : G-H$.

For then $\frac{B}{A} - \frac{C}{A} = \frac{G}{F} - \frac{H}{F}$ (Ax. 3); and $\frac{B-C}{A} = \frac{G-H}{F}$.

Cor. 2. *The ſame things ſuppoſed as in Cor.* 1, *then* $B : C :: G : H$.

For $\frac{B}{A} = \frac{G}{F}$, and $\frac{C}{A} = \frac{H}{F}$, whence $B : C :: \left(\frac{B}{A} \text{ or } \frac{G}{F} : \frac{C}{A} \text{ or } \frac{H}{F} ::\right) G : H$ (Pr. V. and Cor. 1).

PROP. XV.

If there be two ranks of quantities; and it be, in theſe two ranks, as the firſt to the ſecond, ſo is the firſt to the ſecond; and as the ſecond to the third, ſo the ſecond to the third; and ſo on: then will the firſt be to the laſt, as the firſt to the laſt, in the two ranks. If A, B, C, D; *and* F, G, H, I, *are two ranks; and it be,* $A : B :: F : G$, *and* $B : C :: G : H$, *and* $C : D :: H : I$; *then* $A : D :: F : I$.

For $\frac{A}{B}=\frac{F}{G}$, and $\frac{B}{C}=\frac{G}{H}$, and $\frac{C}{D}=\frac{H}{I}$ (Def. 4); therefore $\frac{ABC}{BCD}=\frac{FGH}{GHI}$ (Ax. 4), or $\frac{A}{D}=\frac{F}{I}$; that is, $A : D :: F : I$.

PROP. XVI.

If two or more rows of quantities are reſpectively proportional; the like terms are proportional, in any two rows.

If $A : B : C : D :: P : Q : R : S$. *Then* $B : D :: Q : S$, *&c.*

Quantities are *reſpectively proportional*, when in the ſeveral rows, the firſt term is to the firſt, the ſecond to the ſecond, the third to the third, &c. in the ſame proportion. And *like terms* are thoſe that are alike ſituated in all the rows; as the third term and the third, the fourth and the fourth, &c.

For ſince $B : C :: Q : R$, and $C : D :: R : S$, therefore $B : D :: Q : S$ (Prop. XV); and ſo of others.

Or thus.

If theſe are reſpectively proportional,

$$\begin{array}{l} A : B : C : D : E :: \\ F : G : H : I : K :: \\ L : M : N : O : P :: \\ Q : R : S : T : V :: \end{array}$$

then $A : D :: Q : T$; and ſo of others.

For A : B : : Q : R, and B : C : : R : S, and C : D : : S : T. Therefore A : D : : Q : T. In like manner G : K : : R : V, and A : E : : L : P, and B : E : : R : V, *&c.* all the ways they can be thus compared.

PROP. XVII.

If there be two ſets of quantities; and if it be as the firſt to the ſecond (in the firſt ſet), ſo the laſt but one to the laſt (in the ſecond ſet); and as the ſecond to the third, ſo the laſt but two, to the laſt but one; and ſo on. Then the firſt will be to the laſt (in the firſt ſet), as the firſt to the laſt (in the ſecond ſet).

Firſt ſet A, B, C.
Second ſet F, G, H.

If A : B : : G : H, *and* B : C : F : G, *&c.*
then A : C : : F : H.

For $\frac{A}{B} = \frac{G}{H}$, and $\frac{B}{C} = \frac{F}{G}$ (Def. 4, 2); therefore $\frac{AB}{BC} = \frac{GF}{HG}$ (Ax. 4), or $\frac{A}{C} = \frac{F}{H}$, and A : C : : F : H.

PROP. XVIII.

If there be four proportional quantities in one rank, and four more in another; and ſeveral ſuch ranks; then the products of the like terms will be proportional.

If A : B : : C : D,
and F : G : : H : I,
and P : Q : : R : S,
then AFP : BGQ : : CHR : DIS.

For $\frac{A}{B} = \frac{C}{D}$, and $\frac{F}{G} = \frac{H}{I}$, and $\frac{P}{Q} = \frac{R}{S}$ (Def. 4), therefore $\frac{AFP}{BGQ} = \frac{CHR}{DIS}$ (Ax. 4), or AFP : BGQ : : CHR : DIS.

Cor. 1. *If* $A : B :: C : D$,
and $B : P :: H : I$,
and $P : Q :: R : S$, *&c.*
then $A : Q :: CHR : DIS$.

For $ABP : BPQ :: A : Q :: CHR : DIS$.

Cor. 2. *The ſame things ſuppoſed with two ranks of proportionals, the quotients of the like terms will be proportional.* $\frac{A}{F} : \frac{B}{G} :: \frac{C}{H} : \frac{D}{I}$.

For $AD = BC$, and $FI = GH$ (Prop. XII); therefore $\frac{AD}{FI} = \frac{BC}{GH}$ (Ax. 5); therefore $\frac{A}{F} : \frac{B}{G} :: \frac{C}{H} : \frac{D}{I}$ (Cor. 1. Prop. XII).

Cor. 3. *The like powers, or the like roots of proportional quantities, will be proportional. If* $A : B :: C : D$, *then* $A^n : B^n :: C^n : D^n$, *and* $\sqrt[n]{A} : \sqrt[n]{B} :: \sqrt[n]{C} : \sqrt[n]{D}$: *n being any number.*

This is plain, by ſuppoſing A, F, P all equal; as alſo B, G, Q; and C, H, R; and alſo D, I, S.

PROP. XIX.

If between any two quantities propoſed, there be interpoſed any number of terms; the proportion of the firſt to the laſt, is compounded of the firſt to the ſecond, the ſecond to the third, and ſo on to the laſt. Suppoſe A, B, C, D, E, F.

The proportion of A *to* F, *is compounded of* A *to* B, B *to* C, C *to* D, D *to* E, *and* E *to* F.

For $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} \times \frac{E}{F}$ or $\frac{ABCDE}{BCDEF} = \frac{A}{F}$, all the intermediate terms deſtroying one another, in the dividend and diviſor.

PROP. XX.

In a ſeries of quantities in geometrical progreſſion, $A : B : C : D : E : F : G \div$; *the product of the extremes is equal to the product of any two means, equally diſtant from the extremes:* $AG = BF = CE$, &c.

For ſince $A : B :: F : G$ (Def. 10); therefore $AG = BF$ (Prop. XII). And ſince $B : C :: E : F$; therefore $CE = (BF =)\ AG$, and ſo on.

Cor. *Hence the product of the extremes, is equal to the ſquare of the middle term; when the number of terms is odd.*

PROP.

PROP. XXI.

If, out of a ſeries of quantities in geometrical progreſſion, there be taken any ſeries of equidiſtant terms; that ſeries will alſo be in geometrical progreſſion.

If A : B : C : D : E : F : G : H : I : K : L : M, *in* ∺,
then B : E : H : L *are alſo* ∺.

For $\frac{B}{C} = \frac{C}{D} = \frac{D}{E} = r$, and $\frac{BCD}{CDE} = r^3 = \frac{B}{E}$ (Ax. 4). Alſo $\frac{E}{F} = \frac{F}{G} = \frac{G}{H} = r$, and $\frac{EFG}{FGH}$ or $\frac{E}{H} = r^3$; alſo $\frac{H}{I} = \frac{I}{K} = \frac{K}{L} = r$, and $\frac{HIK}{IKL}$ or $\frac{H}{L} = r^3$, *&c.* Therefore $\frac{B}{E} = \frac{E}{H} = \frac{H}{L}$ *&c.* (Ax. 8); and B : E : H : L *&c.* are ∺ (Def. 10).

PROP. XXII.

If there be a ſeries of quantities in geometrical progreſſion, A : B : C : D : E : F, &c. ∺; *their differences will alſo be in the ſame geometrical progreſſion,* A : B : : A — B : B — C : C — D, *&c.*

For ſince A : B : : B : C : : C : D, *&c.* (Def. 10); therefore A : A — B : : B : B — C : : C : C — D *&c.* (Prop. XIII). And A : B : : A — B : B — C, and B : C : : B — C : C — D (Prop. IV). That is, A : B : C, *&c.* : : A — B : B — C : C — D, *&c.*

Cor. *The ſecond, third, fourth differences,* &c. *ſhall alſo be in the ſame geometrical progreſſion.*

PROP. XXIII.

If there be a ſeries of quantities in geometrical progreſſion; the ratio of the firſt, to the ſecond, third, fourth, &c. *is in the ſimple, duplicate, triplicate,* &c. *ratio of the firſt to the ſecond, reſpectively. If* A : B : C : D : E, &c. *then* $\frac{A}{B}=\frac{A}{B}$, $\frac{A}{C}=\frac{AA}{BB}$, $\frac{A}{D}=\frac{A^3}{B^3}$, $\frac{A}{E}=\frac{A^4}{B^4}$, &c.

For $\frac{A}{B}=\frac{B}{C}=\frac{C}{D}=\frac{D}{E}$, *&c.* (Def. 10). And $\frac{A}{C}=\frac{A}{B}\times\frac{B}{C}=\frac{AA}{BB}$ (Def. 13), $\frac{A}{D}=\frac{A}{B}\times\frac{B}{C}\times\frac{C}{D}=\frac{A^3}{B^3}$; alſo $\frac{A}{E}=\frac{A}{B}\times\frac{B}{C}\times\frac{C}{D}\times\frac{D}{E}=\frac{A^4}{B^4}$; *&c.*

PROP. XXIV.

If A, B, C, D, E, *&c. be a ſet of quantities in geometrical progreſſion, whoſe differences are infinitely ſmall; and n any number; then it will be,* $A^n : A^n - B^n :: A : n\times\overline{A-B}$.

Since the differences are infinitely ſmall, they will be (nearly) equal, $\overline{A-B}=\overline{B-C}=\overline{C-D}$, *&c.* and $A-C=\overline{A-B}+\overline{B-C}=2\times\overline{A-B}$; $A-D=3\times\overline{A-B}$; $A-E=4\times\overline{A-B}$, *&c.* But $A^2 : B^2 :: A : C$, and $A^3 : B^3 :: A : D$, *&c.* (Prop. XXIII); then

$A^2 : A^2-B^2 :: A : A-C=2\times\overline{A-B}$ (Pr. XII),
alſo $A^3 : A^3-B^3 :: A : A-D=3\times\overline{A-B}$.
and $A^n : A^n-B :: A^n : n\times\overline{A-B}$.

PROP. XXV.

In a rank of quantities in geometrical progreſſion, $A : B : C : D : E$, *whoſe number is* n; *and the ratio* $r = \frac{A}{B}$; *the laſt term* $(E) = \frac{A}{r^{n-1}}$ or $\left.\overline{\frac{B}{A}}\right|^{n-1} \times A$.

For $\frac{A}{B} = r$, or $A = Br$, $B = Cr$, $C = Dr$, $D = Er$.

And $A = Br = Crr = Dr^3 = Er^4$.

Therefore $B = \frac{A}{r}$ the 2d term.

$C = \frac{A}{rr}$ the 3d term.

$D = \frac{A}{r^3}$ the 4th term.

$E = \frac{A}{r^4}$ the 5th term.

And in general the n^{th} term $= \frac{A}{r^{n-1}}$.

PROP. XXVI.

In a rank of quantities in geometrical progreſſion, $A : B : C : D : E$, *whoſe number is* n, *and common ratio* $r = \frac{A}{B}$; *the ſum of all the terms is,* $\frac{AA - BE}{A - B} = \frac{BE - AA}{B - A}$.

For $A : B :: B : C :: C : D :: D : E$ (Def. 10). And $A : B ::$ ($r : 1 ::$) $A + B + C + D : B + C + D + E$, (Prop. X); that is, (putting $S =$ ſum), $A : B :: S - E : S - A$. Therefore $SA - AA = BS - BE$ (Prop. XII); and $SA - SB = AA - BE$, or $SB - AS = BE - AA$ (Ax. 2, 3); therefore $S = \frac{AA - BE}{A - B}$ or $\frac{BE - AA}{B - A}$ (Ax. 5).

Cor. 1. *The ſum of the terms* $= A + \frac{A - E}{A - B}B$, or $A + \frac{E - A}{B - A}B$.

For $A + \frac{A - E}{A - B}B = \frac{AA - AB + AB - BE}{A - B} = \frac{AA - BE}{A - B}$, &c.

Cor. 2. *In a decreaſing geometrical progreſſion, the ſum of all the terms* $= \frac{rA - E}{r - 1}$.

For ſince $r : 1 :: S - E : S - A$. Therefore $S - E = rS - rA$ (Prop. XII); and $rS - S = rA - E$ (Ax. 2, 3); whence $S = \frac{rA - E}{r - 1}$ (Ax. 5).

Cor. 3. *In an increaſing geometrical progreſſion; put* $R = \frac{B}{A}$, *then the ſum of the terms* $= \frac{R^n - 1}{R - 1}A$.

For $B = RA$, $C = RB = R^2A$, $D = RC = R^3A$, $E = rD = r^4A$, or $E = r^{n-1}A$. But $1 : R :: S - E : S - A$, and $S - A = RS - RE$ (Prop. XII), and $RS - S = RE - A = r^nA - A$ (Ax. 2, 3); whence $S = \frac{r^nA - A}{r - 1}$ (Ax. 5).

PROP.

PROP. XXVII.

In an infinite decreaſing geometrical progreſſion, A : B : C : D : E ∺ &c. *Put the ratio* $\frac{A}{B} = \frac{m}{n}$; *then the ſum of all the terms* ad infinitum $= \frac{AA}{A-B}$ *or* $\frac{mA}{m-n}$.

For the ſum $= \frac{AA - BE}{A - B}$ (Prop. XXVI); but when the progreſſion is infinitely continued, the laſt term E is o, and then the ſum becomes $\frac{AA}{A-B}$. Alſo (by Cor. 2. Prop. XXVI), the ſum $= \frac{rA - E}{r - 1}$ becomes $\frac{rA}{r-1} = \frac{\frac{m}{n}A}{\frac{m}{n} - 1} = \frac{mA}{m-n}$.

SECT. III.

General Proportions.

Definition and Notation.

IF A, B, C, D, *&c.* be any variable quantities, and a, b, c, d, &c, other values thereof; and if they be ſo dependent on one another, that when A is increaſed or diminiſhed to a; B, C, D, *&c.* become b, c, d, &c.

Then $A \propto B$, ſignifies that A is directly as B, or that $A : a :: B : b$.

Likewiſe $A \propto \frac{1}{C}$, denotes that A is reciprocally as C, or that $A : a :: \frac{1}{C} : \frac{1}{c}$.

Alſo $A \propto \frac{BC}{D}$, ſignifies that A is directly as B and C, and reciprocally as D, or that $A : a :: \frac{BC}{D} : \frac{bc}{d}$.

And if $AB \propto \frac{C}{D}$, the product of A, B is directly as C, and reciprocally as D; or $AB : ab :: \frac{C}{D} : \frac{c}{d}$.

And on the contrary, if $A : a :: B : b$, then $A \propto B$, *&c.*

PROP. I.

If one quantity A *is as a second* B; *then, on the contrary, the second* B *is as the first* A. *If* $A \propto B$, *then* $B \propto A$.

For $A : a :: B : b$ (Def.).
Therefore $B : b :: A : a$; that is, $B \propto A$ (Def.).

PROP. II.

If one quantity A *is as a second* B, *and the second* B *as the third* C, *and the third* C *as a fourth* D, &c. *then the first* A *is as the last* D. *If* $A \propto B \propto C \propto D$, *then* $A \propto D$.

For $A : a :: B : b$,
and $B : b :: C : c$,
and $C : c :: D : d$ (Def.)
therefore $A : a :: D : d$ (Prop. I. Sect. II).
therefore $A \propto D$ (Def.).

Cor. *If one quantity* A *is as a second* B, *and the second* B *reciprocally as a third* C. *Then the first* A *is reciprocally as the third* C. *If* $A \propto B \propto \frac{1}{C}$, *then* $A \propto \frac{1}{C}$.

For $A : a :: B : b : \frac{1}{C} : \frac{1}{c}$; and $A \propto \frac{1}{C}$ (Def.).

PROP. III.

If one quantity A *be as a ſecond* B, *and alſo as a third* C; *then the firſt* A *will be as the ſum or difference of the ſecond and third,* C *and* D. *If* $A \propto B \propto C$, *then* $A \propto B + C$, *or* $A \propto B - C$.

For $A : a :: B : b :: C : c$. Therefore $A : a :: B + C : b + c$, or $A : a :: B - C : b - c$ (Prop. X. Sect. II). And $A \propto B \pm C$.

PROP. IV.

Either ſide of a general proportion, may be multiplied or divided by any given quantity. If $A \propto B$, *then* $A \propto nB$, *or* $A \propto \frac{B}{n}$.

For $A : a :: B : b :: nB : nb$ (Prop. V. Sect. II) $:: \frac{B}{n} : \frac{b}{n}$ (Cor. 1. ibid.).

PROP. V.

If both ſides of a general proportion be multiplied or divided by any variable quantity, they will ſtill be proportional. If $A \propto B$, *and* C *a variable quantity, then* $AC \propto BC$.

For $A : a :: B : b$ (Def.). And $CA : ca :: CB : cb$ (Prop. VI. Cor. 1); that is, $CA \propto CB$.

Alſo $A : a :: B : b$; and $\frac{A}{C} : \frac{a}{c} :: \frac{B}{C} : \frac{b}{c}$ (Cor. 2. Prop. VI); that is, $\frac{A}{C} \propto \frac{B}{C}$.

Cor. 1. *If* $Q \propto BC$, *then* $\frac{Q}{B} \propto C$, *and* $\frac{Q}{BC}$ *is a given quantity, or always the ſame.*

For $\frac{Q}{BC}$ is as 1, an invariable quantity.

Cor. 2. *If* $A \propto \frac{1}{B}$, *then* $B \propto \frac{1}{A}$.

For $AB \propto 1$ (Prop. V), $\frac{AB}{A}$ or $B \propto \frac{1}{A}$ (ibid.).

PROP. VI.

Inſtead of any quantity in one ſide of a general proportion, one may ſubſtitute any other quantity proportional thereto. If $A \propto BC$, *and* $C \propto D$; *then* $A \propto BD$.

For ſince $C \propto D$, $BC \propto BD$ (Prop. V); whence $A \propto BD$ (Prop. II).

PROP. VII.

If the two ſides of one general proportion, be multiplied or divided by the two ſides of another general proportion; they will ſtill be proportional. If $A \propto B$, *and* $C \propto D$; *then* $AC \propto BD$, *and* $\frac{A}{C} \propto \frac{B}{D}$.

For $A : a :: B : b$, and $C : c :: D : d$, therefore $AC : ac :: BD : bd$ (Prop. XVIII. Sect. II); that is, $AC \propto BD$.

And $\frac{A}{C} : \frac{a}{c} :: \frac{B}{D} : \frac{b}{d}$ (ibid. Cor. 1); that is, $\frac{A}{C} \propto \frac{B}{D}$.

Cor. 1. *The equal powers or roots of both ſides of any general proportion, will ſtill be proportional. If* $A \propto B$, *then* $A^2 \propto B^2$, $A^3 \propto B^3$, $\sqrt[2]{A} \propto \sqrt[2]{B}$, *&c.*

This is plain by putting $C = A$, and $D = B$, &c.

Cor. 2. *If* $A \propto B \propto C$, *then* $AA \propto BC$.

PROP. VIII.

If any quantity Q be as the product of ſeveral others A, B, *&c. then if* B, *&c. be given,* $Q \propto A$; *and if* A, &c. *be given,* $Q \propto B$.

For by Prop. IV. ſince $Q \propto AB$, and B given, $Q \propto A$. And if A be given, $Q \propto B$ (ibid.).

Cor. *If any variable quantity Q depends on ſeveral others* A, B; *and if* $Q \propto A$, *when* B *is invariable*; *and* $Q \propto B$, *when* A *is invariable*; *then* $Q \propto AB$, *when all are variable.*

PROP. IX.

Any general proportion may be turned into an equation, by multiplying one ſide by a proper homologous quantity.

If $A \propto BC$, *then* $A = n \times BC$. *n being ſome given quantity.*

For ſince $A \propto BC$, therefore $A : a :: BC : bc$ (Def.); and $A \times bc = a \times BC$ (Prop. XII. Sect. II); and $A = \frac{a}{bc} \times BC$, therefore $n = \frac{a}{bc}$ the quantity aſſumed for a multiplier.

Or if $mA = BC$, it will be found that $\frac{bc}{a} \times A = BC$, or that $m = \frac{bc}{a}$.

PROP. X. *Problem.*

Any general proportion being given, A $\propto \frac{B^2C}{D}$; *to find the proportion any one has to the reſt.*

This is done by help of the foregoing propoſitions.

Since A $\propto \frac{B^2C}{D}$;

Multiply by D, then AD $\propto$ B^2C (Prop. V).

Divide by A, then D $\propto \frac{B^2C}{A}$ (Prop. V).

Divide (AD $\propto$ B^2C) by B^2, and then C $\propto \frac{AD}{B^2}$ (Prop. V).

Divide (AD $\propto$ B^2C) by C, and then $B^2 \propto \frac{AD}{C}$ (Prop. V).

Extract the ſquare root, B $\propto \sqrt{\frac{AD}{C}}$.

And the ſame may be done by aſſuming a given quantity *m*, and making $mA = \frac{B^2C}{D}$; and the foregoing proceſs is the ſame as in the reduction of algebraic equations.

F I N I S.

Emerson, William

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THE

PREFACE.

HAVING in the firſt volume treated of arithmetic, which is one of the main pillars of the mathematics; I come now to geometry, which is the other pillar, on which theſe ſciences are ſupported. On theſe two foundations, all the other branches are built; and from them they derive their whole ſtrength and evidence. And theſe two ſciences are eſſentially different; the former conſiders numbers, without any regard to extenſion; the latter conſiders extenſion, without any regard to numbers. And both of them treat their particular ſubjects in the moſt abſtract manner.

2-27-39 HCm

Geometry is of ſo excellent a nature and of ſuch extenſive uſe, that it lays the foundation of all the rules to work by, in the common affairs of life, without which we could do nothing. For inſtance, the diſtances of places, or remote objects, and their ſituation in reſpect of one another; cannot be had without meaſuring, and the rules of geometry. The drawing of maps or charts can only be done by geometry. The meaſuring and dividing of lands, to give every man his due ſhare, cannot be performed, without meaſuring certain figures, and finding their contents. Houſes and towns cannot be built without knowing the figures and dimenſions *thereof. Without this art, no place can be fortified, to reſiſt the attacks of an enemy. Tradeſmen muſt be acquainted with the meaſures of length and capacity. Joiners, maſons, &c. muſt underſtand how to form their materials into proper figures, where there will be frequent occaſion for parallel and perpendicular lines. And the figures they have perpetually to deal with, are triangles, ſquares, parallelograms, circles, &c. and ſuch ſolids as pyramids, cones, cubes, priſms, ſpheres, &c. the nature of which can only be known from geometry. The dimenſions and areas of plane figures, the contents of ſolid bodies, cannot be had without it. So that geometry gives life and ſpirit to all arts.*

Geometry examines the nature of all figures, compares them together, and finds out their properties. It is a key to all the other branches. The elements of plane geometry, are likewiſe the foundation of the higher geometry, relating to all ſorts of curve lines, their nature and properties; and is a neceſſary introduction to the knowledge of them.

Geometry is a ſcience inexhauſtible, and which knows no bounds. In it there is always room left for the diſcovery of new theorems. It is alſo a moſt excellent logic, teaches men how to reaſon truly, and accuſtoms the mind to a habit of cloſe and ſtrict reaſoning.

The ſcience of geometry is certainly very old; for look as far back as we will, we ſhall always find men who have been profeſſors and encouragers of geometry, and the value the ancients ſet upon it, may be known from this famous motto of Plato *ſet over the door of his academy,* οὐδεὶς ἀγεωμέτρητος εἰσίτω. *Some of the principal among them who ſtudied it were*, Thales, Pythagoras, Plato, Ariſtotle, Euclid, Archimedes, Appollonius, Ptolomy, *and many more. But we are not to ſuppoſe that in theſe ancient times, this ſcience was any thing near the perfection it is now in: but in ſucceeding ages, men of great genius, by their ſtudy and induſtry, by degrees added new improvements; till at laſt it arrived at the pitch we now ſee it. So that we need not wonder that* Euclid, *or even* Archimedes, *have taken round-about methods in demonſtrating many of their propoſitions, which are now done vaſtly ſhorter and clearer. For it cannot be denied, that* Euclid'*s elements abound with a great many trifling propoſitions, which are of no other uſe but to demonſtrate, in his way, the propoſitions that follow after. But they are diſpoſed in no proper order or method. For he frequently treats of different ſubjects, promiſcuouſly together, in the ſame place; without any regard to the nature of things, or their connection with one another. And as often, has the ſame ſubject to conſider in different places; which can breed nothing but confuſion. But there are likewiſe a great many propoſitions in the preſent ſyſtem of geometry, which theſe ancient mathematicians knew nothing of; and which are equally uſeful with thoſe of* Euclid.

One method of demonſtration, which Euclid *and the ancients frequently make uſe of, is* reductio ad abſurdum, *which is generally ſhorter than the direct method, and equally certain. For it is an axiom in logic, that* that ſuppoſition muſt needs be true, which deſtroys the contrary ſuppoſition. *But though it be equally true, yet it gives not that ſatisfaction to the mind, which a poſitive proof gives.*

It is a common practice among geometers, after a propoſition is proved, for them to prove the reverſe of it. But this in many caſes is needleſs and impertinent. For where the eſſential property of a ſubject is found; there, moſt certainly, you will find that ſubject, without farther inquiry. For example, when it is proved to be the property of parallel lines, when cut by a third line, to make the alternate angles equal; or the ſum of the internal angles equal to two right angles: it is ſuperfluous to prove, that when the alternate angles are equal, or the ſum of the internal angles equal to two right ones, that theſe lines are parallel; becauſe it was proved before to be the abſolute right and property of parallel lines. Likewiſe when it is proved to be the diſtinguiſhing property of a right-angled triangle, that the ſquare of the hypothenuſe is equal to the ſum of the ſquares of the two ſides. It need not be proved, that when theſe ſquares are equal, the angle is right. In ſuch caſes, there needs, at moſt, nothing but an illuſtration, and then this method (reductio ad abſurdum) *is very properly applied.*

There are alſo many propoſitions in geometry, which are convertible; that is, where the property or predicate may become the ſubject; and the ſubject, the predicate; being of equal extent. And here a deal of labour might be ſaved in demonſtrating the propoſition both ways. For inſtance, when *the two ſides of a triangle are equal, it may be proved, that the two oppoſite angles are equal. Or when the two angles of a triangle are equal, it may be proved, that the oppoſite ſides are equal. But it need not be proved both back and forward. And here can want nothing but the application of the former rule* (reductio ad abſurdum), *to illuſtrate the reverſe. But mathematicians had rather prove too much than too little; they had rather have ſomething* ex abundanti, *than be defective. Though for my own part, I have often ſaved myſelf that ſuperfluous labour.*

To give ſome account of the method wherein I have handled this ſubject; it is in ſhort this. The firſt book treats of right lines. The ſecond of triangles. The third of polygons. The fourth of the circle. The fifth of planes. The ſixth of ſolids. The ſeventh of the ſphere. The eighth is geometrical problems. This is the method I have choſen to digeſt theſe things in, as being agreeable to the nature of the ſubject, and the mutual dependance of the ſeveral parts upon one another. The laſt book contains a collection of the moſt uſeful geometrical problems. I have ſpent but little time in demonſtrating them, as moſt of them do not need it, being perſuaded that they who underſtand the elements, will eaſily perceive their evidence, without any more words. They that would ſee more problems of this kind, may conſult the writers of practical geometry.

W. Emerſon.

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THE ELEMENTS OF GEOMETRY.

DEFINITIONS.

1. GEOMETRY is a ſcience which teaches and demonſtrates the properties, affections, and meaſures of all ſorts of magnitude. FIG.

2. *Magnitude* is continued quantity, or any thing that is extended; as a line, ſurface, or ſolid.

3. A *point* is that which has no parts.

4. A *line* is a length without breadth or thickneſs.

Cor. *The extremes of a line are points.*

5. A *right line* is that which lies evenly, or in the ſame direction, between two points A, B. A *curve line* continually changes its direction. 1.

Cor. *Hence there can only be one ſpecies of right lines, but there is infinite variety in the ſpecies of curves.*

6. *Parallel lines* are thoſe which are always at the ſame diſtance from one another, as AB, CD. 2.

7. An *angle* is the inclination of two lines, to one another, meeting in a point, called the *angular point.* When it is formed by two right lines, it is a *plain angle*, as A; if by curve lines, it is a *curvilineal angle.* 3.

FIG. 8. A *right angle* is that which is made by one right line AB falling upon another CD, and
4. making the angles on each ſide equal, ABC = ABD; ſo that AB does not incline more to one ſide than another: AB is called a *perpendicular*. All other angles are called *oblique angles*.

9. An *obtuſe angle* is greater than a right angle,
5. as R.

10. An *acute angle* is leſs than a right angle, as S.

11. *Contiguous angles*, are thoſe made by one line falling upon another, and joining to one another, as R, S.

12. *Oppoſite angles*, are thoſe made on contrary ſides of two lines interſecting one another, as
6. A, B.

13. A *ſurface* is that magnitude which hath only length and breadth.

Cor. *The extremes or limits of a ſurface are lines.*

14. A *plane* is that ſurface which lies perfectly even between its extremes; or in which, right lines any way drawn, do coincide.

15. A *plain figure*, is a plain ſurface, bounded on all ſides by one or more lines.

16. A *right-lined figure*, is a plain figure, bounded with right lines only.

Cor. *Every right-lined figure has as many angles as ſides.*

17. A *ſolid* is a magnitude extended every way, or which has length, breadth, and depth.

Cor. *The terms or extremes of a ſolid, are ſurfaces.*

18. The *ſquare of a right line* is the ſpace included by four right lines equal to it, ſet perpendicular to one another.

19. The *rectangle of two lines* is the ſpace included by four lines equal to them, ſet perpendicular to one another, the oppoſite ones being equal.

20. *Commenſurable magnitudes*, are ſuch as may be meaſured by one and the ſame meaſure.

21. *Incommenſurable magnitudes*, are ſuch as have no common meaſure.

22. *Rational magnitudes*, are thoſe that are expreſſed by a rational number, or by one that includes not a ſurd.

23. *Irrational magnitudes*, are ſuch as are denoted by a ſurd, as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, *&c.*

AXIOMS or MAXIMS.

1. Things equal to the ſame thing, are equal to one another.
2. The whole is equal to all its parts taken together.
3. If equal things be added to equal things, the wholes will be equal.
4. If equal things be taken away from equal things, the remainders will be equal.
5. If equal things be equally multiplied, the products will be equal.
6. If equal things be equally divided, the quotients will be equal.
7. All right angles are equal to one another.
8. Thoſe magnitudes are equal, which being applied, exactly agree or coincide with one another.

POSTULATES.

1. Between any two points a right line may be drawn.
2. That a right line or plane may be produced as far as we pleaſe.
3. That a circle may be deſcribed from any center at any diſtance. See Book IV. Def. 1.
4. That any magnitude being given, an equal magnitude may be made.

5. That any magnitude may be ſo often multiplied, as to exceed any magnitude of the ſame kind.

6. That any magnitude may be divided into as many equal parts as we pleaſe.

Explanation of Characters.

$+$	added to, being the ſign of addition.
$-$	ſubtracted from, the ſign of ſubtraction.
$\times$	multiplied by.
$\div$	divided by.
$=$	equal to.
$::$	the mark of proportion.
$\because\!\!\div$	geometrical progreſſion.
$\backsim$	difference.
$\square$	ſquare.
▭	rectangle.
$\sqrt{\ }$	ſquare root.
$\sqrt[3]{\ }$	cube root.
A^2	A ſquared; alſo $\overline{AB}^2$ is AB ſquared.
A^3	A cubed; and AB^3 is AB cubed.
$\angle$	an angle.
$\parallel$	parallel.
$\perp$	perdendicular.

Sometimes one letter denotes a line; but if a line is expreſſed by two letters, as AB, then the letters A, B denote the extreme points of that line.

When one letter denotes an angle, it is ſuppoſed to ſtand at the angular point; but if three letters expreſs the angle, the middle one is at the angular point; the other two in the ſides.

When three letters ſtand for a rectangle, as ABC, it ſignifies AB × BC; where AB, BC are the ſides. Or when four letters ſtand for a rectangle, as AB × CD; AB and CD are the ſides.

The citations are thus to be underſtood; the firſt number denotes the Prop. the ſecond the Book. When proportion is referred to, it ſignifies geometrical proportion.

BOOK I.

Of Angles, and Right Lines, and their Rectangles.

PROP. I.

If to any point C *in a right line* AB, *ſeveral other right lines* DC, EC *are drawn on the ſame ſide; all the angles formed at the point* C, *taken together, are equal to two right angles,* $ACD + DCE + ECB$ $=$ *two right angles.* FIG. 7.

FOR ſuppoſe PC to be perpendicular to AB, then ſince ACP and PCB $=$ two right angles, (Def. 8); and theſe angles ACD, DCE, ECB occupy the ſame angular ſpace; therefore they are all equal to two right angles (Ax. 2).

Cor. 1. *All the angles made about one point in a plane, being taken together, are equal to four right angles.*

Cor. 2. *If all the angles at* C, *on one ſide of the line* AB, *happen to be equal to two right angles; then* ACB *is a ſtraight line.*

PROP. II.

If two right lines, AB, CD, *cut one another; the oppoſite angles* E *and* G *will be equal.* 8.

For $AEC + E =$ two right angles (Prop. I), and $AEC + G =$ two right angles (ibid.); therefore $AEC + E = AEC + G$ (Ax. 1), and $E = G$ (Ax. 4). After the ſame manner $AGC = BGD$.

FIG. 8. Cor. *If* AB *is a right line, and* CEB *happen to be equal to* AGD, *or* E *equal to* G; *then* CD *is a right line.*

PROP. III.

9. *A right line* AB, *which is perpendicular to one of two parallels* FH, *is perpendicular to the other* DC.

For ſuppoſe the end HC of the figure CBAH, be raiſed up, and turned over the line AB, ſo that HC may fall towards FD, the line AB remaining fixed. Then ſince the ∠BAH = BAF (Ax. 7), therefore the line AH will fall upon AF, and let the line BC fall on the line B*d*. Draw the line *d*DF perpendicular to HF. Now ſince FH, DC are parallels; therefore the diſtances BA, DF, and *d*F (or CH) are all equal (Def. 6); therefore the points D, *d* muſt coincide; and therefore the line B*d* coincides with BD. Therefore ∠ABC = ABD = a right angle (Def. 8).

Cor. 1. *Hence two lines* FH, DC, *perpendicular to the ſame line* AB, *are parallel.*

Cor. 2. *Hence the ſegments of two parallels, intercepted between two perpendiculars* AB, HC, *are equal,* AH = BC.

For ſince the angles at A, H, B, C are right, therefore the two lines AB, HC, interſecting AH, and being both perpendicular thereto, are parallel (Cor. 1); and therefore AH = BC (Def. 6).

PROP. IV.

10. *If a right line* CG, *interſect two parallels* AD, FH; *the alternate angles,* ABE, *and* BEH, *will be equal.*

Let AE, BH be perpendicular to AD, and FH. Then ſince AE = BH (Def. 6), and AB = EH (Prop. III. Cor. 2), and the angles at A and H right; therefore if the figure EHB be laid upon BAE, the ∠H upon A, and HE upon AB; and conſequently HB will fall upon AE; and the whole figure EHB coincides with the figure BAE, and the angle HEB with EBA, and conſequently theſe angles are equal. Likewiſe the angles DBE and FEB will be equal, being the remainders to two right angles (Ax. 4). FIG. 10.

Cor. 1. *The external angle* CBD, *is equal to the internal angle on the ſame ſide* BEH.

For CBD = ABE = BEH (Prop. 2).

Cor. 2. *The two internal angles on the ſame ſide are equal to two right angles*; DBE + BEH = *two right angles.*

For EBA = BEH (Prop. IV), and DBE + EBA = two right angles, = DBE + BEH.

Cor. 3. *If the angles* CBD *and* BEH *are equal; or* ABE *and* BEH *equal; or* DBE + BEH *be equal to two right angles; the lines* AD, FH *are parallel.*

For if any angle is greater than is here mentioned, it deſtroys the paralleliſm of the lines AD, FH.

PROP. V.

Two lines drawn between two parallels AB, CD, *making equal angles with either of them, will be equal,* AC = BD. 11.

Draw CF, DH perpendicular to FB, then ſince ∠ACD = BDI; alſo FCD and HDI right angles (Prop. III), the remainders FCA and HDB are equal; and the angles at F and H being right, and FC = HD (Def. 6); therefore if HD be laid on FC, the line DB will fall on CA, and HB on FA, and B on A; therefore DB = CA.

Cor. 1. *If the lines* AC *and* BD *are equal, then the angles* ACD *and* BDI *are equal.*

11. For if one angle was greater, it would make the lines AC, BD unequal.

Cor. 2. *The parts intercepted are equal,* $AB = CD$.

For $FA = HB$, and adding AH, $AH + HB$, or $AB = FA + AH$, or $FH = CD$ (Cor. 2. Prop. III).

Cor. 3. *If two equal and parallel lines* AB, CD, *be joined by two others* AC, BD; *they ſhall alſo be equal and parallel.*

PROP. VI.

12. *Right lines* AB, CD, *parallel to the ſame right line* EF, *are parallel to one another.*

Let GI cut the three lines, then ſince AB, EF are parallel, $AGI = EHI$ (Cor. 1. Prop. IV); and becauſe EF and CD are parallel, $\angle EHI = DIG$ (Prop. IV). Therefore $AGI = DIG$ (Ax. 1), whence AB, CD, are parallel (Cor. 3. Prop. IV).

PROP. VII.

13. *If two lines* AB, BD, *which cut one another, be parallel to two other lines* EC, CH, *which alſo cut one another; they ſhall contain equal angles* $ABD = ECH$.

For produce EC to interſect BD in F; then by reaſon of the parallels AB, EF, $\angle ABD = EFD$ (Cor. 1. Prop. IV); and ſince BD and CH are parallel, $EFD = ECH$ (ibid.); therefore $ABD = ECH$.

PROP. VIII.

FIG. 14.

Two right lines AF, AB *being given; and one of them* AB *be divided into ſeveral parts; the rectangle under the two whole lines, will be equal to all the rectangles contained under the whole line, and the ſeveral ſegments of the other;* $ABGF = ADHF + DEIH + EBGI$.

For let AF be perpendicular to AB, and DH, EI, BG equal to AF, and alſo perpendicular to AB. Then $AD \times AF =$ rectangle ADHF, and $HD \times DE$, or $FA \times DE =$ rectangle DEIH, and $IE \times EB$, or $AF \times EB =$ rectangle EBGI (Def. 19); but the ſum of theſe rectangles fill the ſame ſpace as ABGF, and therefore they are equal (Ax. 8).

Cor. 1. *If both lines be divided into parts, the ſum of the rectangles of all the parts, is equal to the rectangle of the wholes.*

Cor. 2. *If the two given lines be equal; the ſum of the rectangles under the whole and the parts, is equal to the ſquare of the whole.*

PROP. IX.

If a line AC *be divided into two parts at* B; *the rectangle under the whole, and one of the ſegments,* $AC \times BC$, *is equal to the rectangle of the ſegments and the ſquare of the ſaid ſegment,* $AB \times BC + BC^2$. 15.

Suppoſe AF, BE, CD all equal to BC, and perpendicular to AC; then the rectangle $ACDF = AC \times CD = AC \times BC$ (Def. 19); alſo $AB \times BC = AB \times BE =$ rectangle ABEF, and $BC \times CD$ or $BC^2 = BCDF$ (Def. 18). But $ABEF + BCDE$ fill the rectangle ACDF, and therefore they are equal (Ax. 8).

PROP.

PROP. X.

FIG. 16. *If a right line* AC *be divided into two parts* AB, BC; *the ſquare of the whole line is equal to the ſquares of both the parts, and twice the rectangle of the parts,* $AC^2 = AB^2 + BC^2 + 2AB \times BC$.

Let AG, BH, CI be equal to AC, and perpendicular thereto, and AD, BE, CF equal to AB; then FI = BC, *&c.* then ABED is the ſquare of AB (Def. 18), and EFIH is the ſquare of BC; and the figures BF and EG, are the rectangles of BC and BE, and DG and DE; or of AB and BC twice taken (Def. 19). But all theſe fill the ſquare AI, and therefore are equal to it (Ax. 8).

PROP. XI.

16. *The ſquare of the difference of two lines* AC, BC, *is equal to the ſum of their ſquares, wanting twice their rectangle;* $AB^2 = AC^2 + BC^2 - 2AC \times BC$.

For the ſquare AI contains the ſquare AE, the rectangle CH, and rectangle DH; that is, $AC^2 = AB^2 + CH + DH$; and adding FH, $AC^2 + FH = AB^2 + CH + DI$; that is, $AC^2 + BC^2 = AB^2 + 2ACB$, and AB^2 or $\overline{AC - BC}^2 = AC^2 + BC^2 - 2ACB$.

PROP. XII.

16. *The rectangle of the ſum and difference of two lines* AC, AB, *is equal to the difference of their ſquares,* $\overline{AC + AB} \times BC = AC^2 - AB^2$.

For the difference of the ſquares AI and AE is the rectangles $CH + HD = \overline{BH + HG} \times BC = \overline{AC + AB} \times BC$.

PROP. XIII.

The ſquare of the ſum, together with the ſquare of the difference of two lines, is equal to twice the ſum of their ſquares.

Let the lines be A, E. Then
the ſquare of $A+E = A^2+E^2+2AE$ (Prop. X).
the ſquare of $A-E = A^2+E^2-2AE$ (Prop. XI).
then $\overline{A+E}^2+\overline{A-E}^2 = 2A^2+2E^2$ (Ax. 3).

PROP. XIV.

The difference of the ſquares, made of the ſum and difference of two right lines, is equal to four times their rectangle.

For if A, E be the lines, then
$\overline{A+E}^2 = A^2+E^2+2AE.$
$\overline{A-E}^2 = A^2+E^2-2AE.$
difference $=$ $4AE.$

Cor. *The ſquare of the ſum is equal to the ſquare of the difference, together with four times their rectangle.*

◆

BOOK

BOOK II.

Of Triangles.

DEFINITIONS.

1. A *Triangle* is a plain figure bounded by three right lines, called the *ſides* of the triangle.

2. An *equilateral triangle* is that which has three equal ſides.

3. An *equiangular triangle* is that which has three equal angles.

4. An *iſoſceles triangle* is that having two ſides equal.

5. A *right-angled triangle* is that which has a right angle. The ſide oppoſite to the right angle is called the *hypothenuſe.*

6. An *oblique triangle* is that having oblique angles.

7. An *obtuſe angled triangle* has one obtuſe angle.

8. An *acute angled triangle* has three acute angles.

9. A *ſcalenous triangle* has three unequal ſides.

10. *Similar triangles* are thoſe whoſe angles are reſpectively equal, each to each. And *homologous ſides* are thoſe lying between equal angles.

11. *Baſe* of a triangle, is the ſide on which a perpendicular is drawn from the oppoſite angle called the *vertex*; the two ſides, proceeding from the vertex, are called the *legs.*

PROP.

PROP. I. FIG.

In any triangle ABC, *if one ſide* BC *be drawn out; the external angle* ACD *will be equal to the two internal oppoſite angles* A, B. 17.

Draw CE parallel to AB, then the $\angle A = ACE$ (4. 1); alſo the $\angle B = ECD$ (Cor. 1. ibid.); therefore $A + B = ACE + ECD = ACD$ (Ax. 3).

PROP. II.

In any triangle ABC, *the ſum of the three angles is equal to two right angles*, $A + B + C =$ *two right angles*. 17.

For $A + B = ACD$ (Prop. I), and $A + B + C = ACD + ACB$ (Ax. 3) = two right angles (1. I).

Cor. 1. *If two angles in one triangle, be equal to two angles, in another; the third will alſo be equal to the third.*

Cor. 2. *If one angle of a triangle be a right angle, the ſum of the other two will be equal to a right angle.*

Cor. 3. *There can only be one perpendicular drawn, to any line, from a given point.*

PROP. III.

The angles at the baſe of an iſoſceles triangle, are equal $\angle C = B$. 18.

For let AD biſect the angle BAC; then if the triangle DAC be laid upon the triangle DAB; then by reaſon of the equal angles at A, and $AC = AB$, AC will coincide with AB, and C with B, and CD with BD; and therefore $\angle ACD = ABD$.

Cor. 1. *If the angles* B, C *at the baſe be equal, the ſides* AB, AC *are equal.*

Cor. 2. *An equilateral triangle is alſo equiangular; and the contrary.*

Cor.

FIG. Cor. 3. *The line which is perpendicular to the baſe of an iſoſceles triangle, biſects it and the vertical angle.*

Cor. 4. *Only two equal lines can be drawn from a given point to a right line.*

For if $AB = AD = AC$; then $\angle B$ as well as $\angle D = \angle C$, which is abſurd (Prop. I).

PROP. IV.

In any triangle, the greateſt ſide is oppoſite to the greateſt angle, and the leaſt to the leaſt.

19. Let AC be the greateſt ſide, and ſuppoſe $AD = AB$, then the $\angle ADB = ABD$ (Prop. III), but $ADB = DBC + DCB$ (Prop. I); therefore ADB is greater than C; whence ABD is greater than C, therefore much more is ABC greater than C. After the ſame manner it is proved, that ABC is greater than A.

And if AB be the leaſt ſide, C is leſs than ABC; and may be proved in like manner to be leſs than A.

PROP. V.

20. *In any triangle* ABC, *the ſum of any two ſides* BA, AC, *is greater than the third* BC.

Produce the ſide BA, and let $AD = AC$, and draw DC; then ſince $\angle ACD = D$ (Prop. III); therefore BCD is greater than D; and therefore the oppoſite ſide BD is greater than BC, that is, $BA + AC$ is greater than BC.

Cor. 1. *A right line is the ſhorteſt diſtance between any two points.*

21. Cor. 2. *The ſum of two lines* BD, DC, *drawn from two angles to any point within the triangle, is leſs than the two ſides of the triangle;* $BD + DC$ *is leſs than* $BA + AC$, *but contain a greater angle.*

For drawing BDE, then, in the triangle BAE, BE is leſs than $BA + AE$ (Prop. V), add EC, then $BE + EC$ is leſs than $BA + AC$. And in the triangle DEC, DC is leſs than $DE + EC$; add BD, and $BD + DC$ is leſs than $BE + EC$, and much leſs than $BA + AC$.

Alſo $\angle BDC$ is greater than DEC, which is greater than A (Prop. I).

PROP. VI.

If two triangles ABC, *abc, have two ſides and the included angle equal in each; theſe triangles, and their correſpondent parts, ſhall be equal.* 22.

For ſince the $\angle A = a$, and $AB = ab$, alſo $AC = ac$, therefore if A be laid upon *a*, ſo that AB fall upon *ab*, then AC will fall upon *ac*, the point B will coincide with *b*, and C with *c*; therefore the whole triangles coincide. Whence the baſe $CB = cb$, $\angle B = b$, and $C = c$. And the whole triangles are equal.

Cor. *If two triangles* ABC, *abc, have two ſides reſpectively equal; that which has the greater baſe, has the greater oppoſite angle; and the contrary.*

For if the ſides CA, BA intercept a greater baſe BC, the angle at A will be ſo much the wider or greater; and as the angle increaſes, the more of the baſe it intercepts, as is evident.

PROP. VII.

If two triangles ABC *and abc, have two angles and a ſide equal, each to each; the remaining parts ſhall be equal, and the whole triangles equal.* 22.

For ſince two angles are equal, the third will be equal (Cor. 1. Prop. II); therefore if the equal ſides BC and *bc* be laid one upon another, then, by reaſon of the equal angles B and *b*, C and *c*, the

FIG. the ſides BA and *ba* will coincide, as alſo CA and *ca*, and A will fall on *a*; whence all the parts will be equal (Ax. 8).

PROP. VIII.

If two triangles have all their ſides reſpectively equal; all the angles will be equal, and the wholes equal.

23. For if the baſe of one be laid upon the baſe of the other, the other two ſides will coincide, provided the correſpondent ones lie the ſame way. For if you ſay they don't coincide, let one triangle be ABC, the other ABD: then ſince AB, AC are equal to AB, AD (hyp.), and the angle BAD greater than BAC, therefore BD is greater than BC (Cor. Prop. VI); contrary to the hypotheſis.

Cor. 1. *From two points in a right line, as* A *and* B, *two lines equal to* AC, BC *cannot be drawn to any other point* D.

Cor. 2. *Triangles mutually equilateral, are mutually equiangular.*

PROP. IX.

24. *If in two triangles* ABC, *abc*; *two ſides* AC, CB, *of the one, be equal to ac, cb of the other; and an oppoſite angle* A *equal to the correſpondent oppoſite angle a; and the other oppoſite angles* B, *b*, *either both acute or both obtuſe; the remaining parts of the triangles will be equal.*

For if *cab* be laid upon CAB, ſo that *ca* fall upon CA; then ſince the $\angle a = \text{A}$, *ab* will fall upon ABD. And as *c* falls upon C; *cb* will fall upon either CB or CD (Cor. 4. Prop. III); which here will be CB, as the angle at *b* is obtuſe. Therefore the triangles coincide, and all the parts are equal.

PROP.

PROP. X. FIG.

Triangles BCA, BCF, *ſtanding upon the ſame baſe, and between the ſame parallels,* are equal.* 25.

Let CD be parallel to BA, and BE to CF. Then the triangle CBA = ADC (Prop. VI); for BA = CD (5. 1); and CB = AD (Cor. 2. ibid.), and ∠B = D (4. 1). Therefore the triangle BCA = half of BCDA. For the ſame reaſon BCF = BEF = half of CBEF.

Again, the triangles BAE, CDF are equal, having two ſides and the contained angle equal; add the figure BCDE, and then BCDA = BCFE, and their halves BCA = BCF.

Cor. 1. *Triangles of equal baſes and hights are equal.*

For if their baſes be laid upon one another, the angular points of both (by reaſon of their equal hight) will fall in the ſame parallel; and are therefore equal (Prop. X).

Cor. 2. *Every triangle is equal to half the rectangle of its baſe and hight.*

For ſuppoſe CBA to be a right angle, then it was proved that the triangle CBA is half of the rectangle CBAD; and CBF (equal to it), is therefore equal to half that rectangle.

PROP. XI.

Triangles ABC, ABD, *of the ſame hight, are in proportion to one another as their baſes* BC, *and* BD. 26.

Divide BC into any number of equal parts BF, FG, GH, HC; and BD into ſome number of the ſame equal parts, BI, IK, KD. The triangles ABF, AFG, &c. and ABI, AIK, &c. are all equal (Cor. 1. Prop. X); and the triangle ABC contains

FIG. 26. ABF as oft as BC contains BF; alſo ABD contains ABI or ABF as oft as BD contains BI or BF; whence ABF : BF : : ABC : BC : : ABD : BD (Def. 4. Proportion and Cor. 2. Prop. XIV. ibid.).

Cor. 1. *Hence triangles are to one another as their baſes and altitudes.*

It follows from this Propoſition, and Cor. 2. Prop. X. therefore,

Cor. 2. *Triangles of equal baſes, are as their bights.*

PROP. XII.

27. *If a line* DE *be drawn parallel to one ſide* BC, *of a triangle; the ſegments of the other ſides will be proportional;* AD : DB : : AE : EC.

For draw BE, DC; then the triangle DEB = triangle DEC (Prop. X); and triangle ADE : BDE : : AD : BD (Prop. XI); and triangle ADE : CDE : : AE : CE (ibid.); therefore AD : DB : : AE : EC (Prop. I. Proportion).

Cor. 1. *If the ſegments be proportional,* AD : DB : : AE : EC; *then the line* DE *is parallel to the ſide* BC.

For if theſe lines were not parallel, the triangles DEB and DEC would not be equal (Cor. 2. Prop. X); and the ſegments would not be proportional.

Cor. 2. *If ſeveral lines be drawn parallel to one ſide of a triangle, all the ſegments will be proportional.*

Cor. 3. *A line, drawn parallel to any ſide of a triangle; cuts off a triangle ſimilar to the whole.*

For ∠D = B, and ∠E = C (Cor. 1. Prop. IV. I); therefore they are ſimilar (Def. 10).

Cor. 4. *The whole ſides are as the ſegments;* AB : DB : : AC : EC.

For it is AD : DB :: AE : EC (Prop. XII), FIG.
whence AD + DB (AB) : DB :: AE + EC (AC) 27.
: EC (Prop. XIII. Proportion).

PROP. XIII.

In ſimilar triangles, the homologous ſides are pro- 28.
portional; AB : AC :: DE : DF.

In the longer ſide AC make A*f* = DF, the longer ſide. And in the ſhorter ſide AB, make the ſhorter ſide DE = A*e*; and draw *ef*; then the ∠A being ſuppoſed = to D, and the comprehending ſides equal, ∠A*ef* = E, and A*fe* = F (Prop. VI). Therefore A*ef* = B, and A*fe* = C; conſequently *ef* is parallel to BC (Cor. 1. Prop. 4. I); therefore AB : *e*B :: AC : *f*C (Cor. 4. Prop. XII); and AB : AB — *e*B (A*e*) :: AC : AC — *f*C (A*f*), Prop. XIII. Proportion). That is, AB : DE :: AC : DF, or AB : AC :: DE : DF (Prop. IV. Proportion).

And if a triangle was made at the ∠C equal to DFE; it will appear the ſame way, that AC : CB :: DF : FE. Whence AB : CB :: DE : EF (Prop. XV. Proportion).

Cor. *A line* AE *drawn from the oppoſite angle* A, *cuts* 29.
two parallel lines proportionally; BE : EC :: DI : IF.

For BE : DI :: AE : AI :: EC : IF.

PROP. XIV.

If two triangles have one angle equal to one, and the 28.
ſides about the equal angles proportional; *theſe triangles are ſimilar.*

For let ∠D = A, and let the triangle DEF be laid upon ABC; then, by reaſon of the equal angles, the ſides DE, DF will fall upon AB, AC, the points E, F upon *e* and *f*. Then ſince DE (A*e*) : DF (A*f*) :: AB : AC, or A*e* : AB :: A*f* : AC, therefore A*e* :

FIG. 28. eB : : Af : fC (Prop. XIII. Proportion) ; whence ef is parallel to BC, (Cor. 1. Prop. XII) ; and ∠ e or E = B, as alſo f or F = C (Cor. 1. Prop. IV. I). Whence the triangles DEF, ABC are ſimilar (Def. 10).

PROP. XV.

30. *If two triangles have all their ſides reſpectively proportional, theſe triangles are ſimilar*; AB : DE : : BC : EF : : AC : DF.

Let the ∠FEG = B, and EFG = C, then G = A (Cor. 1. Prop. II) ; whence GE : EF : : AB : BC (Prop. XIII) : : DE : EF (hyp.); therefore GF = DE, (Ax. 7. Proportion). Likewiſe GF : EF : : AC : BC : : DF : EF ; therefore GF = DF (Ax. 7. Proportion). Whence the triangles DEF, GEF have all their ſides reſpectively equal ; and are therefore equiangular ; therefore G = D = A, DEF = GEF = B, and GFE = DFE = C.

PROP. XVI.

If two triangles have one angle in each, equal ; and the ſides about the ſecond angles proportional ; and the third angles both of one kind, acute or obtuſe ; theſe triangles are ſimilar.

31. Let ∠A = D, and AB : BC : : DE : EF. Make ∠ABG = DEF, then ∠G = F (Cor. 1. Prop. II.); whence AB : BG : : DE : EF (Prop. XIII.) : : AB : BC, therefore BG = BC, and BCG is an iſoſceles triangle, and AGB is obtuſe, of the ſame kind with DFE ; and ACB is acute, the ſame as DIE ; whence the angles F and G, or I and C, muſt be of the ſame kind, to have the triangles ſimilar.

SCHOLIUM.

This does not always hold good, if the angles B and E are required to be of the ſame kind, inſtead of G and F. For if ABC be acute, ABG will alſo be acute; but ABG is not ſimilar to DEI, nor ABC to DEF; though ABC be ſimilar to DEI, and ABG to DEF. FIG. 31.

PROP. XVII.

Equal triangles, that have one angle equal, have the ſides about the equal angles reciprocally proportional.

Let the oppoſite angles at B be the equal angles, and ABC, DBE, the two equal triangles; then AB : BE : : DB : BC (hyp.). 32.

Draw CE, then AB : BE : : triangle ABC or DBE : triangle CBE (Prop. XI) : : DB : BC.

Cor. 1. *Thoſe triangles are equal, that have the ſides about the equal angles, reciprocally proportional.*

For triangle ABC : CBE : : AB : BE (Prop. XI) : : DB : BC (hyp.) : : triangle DBE : CBE; therefore triangle ABC = DBE (Ax. 7. Proportion).

Cor. 2. *Equal triangles have their baſes and hights reciprocally proportional.*

For each triangle is equal to a right-angled triangle of the ſame baſe and hight (Prop. X); and then the ſides about the right angles, are reciprocally proportional (Prop. XVII).

PROP. XVIII.

Like triangles ABC *and* DEF *are in the duplicate ratio, or as the ſquares of, their homologous ſides,* BC, EF. 33.

Let there be taken BG, ſo that BC : EF : : EF : BG, and draw AG. Then ſince AB : DE : : BC : EF (Prop. XIII) : : EF : BG (Conſtruct.); therefore the triangle ABG = DEF. But ABC : ABG or DEF : : BC : BG (Prop. XI) : : BC^2 : EF^2 (Prop. XXIII. Proportion).

FIG.

PROP. XIX.

34. *Triangles that have one angle equal to one, are to one another in the complicate ratio of the ſides about the equal angles*; $ABC : EBD :: AB \times BC : EB \times BD$.

Draw CE, then CD, AE being ſtraight lines, the angles at B are equal (Prop. II. I). Then triangle $ABC : CBE :: AB : BE$ (Prop. XI), and $CBE : EBD :: CB : BD$ (ibid.); therefore $ABC : EBD :: AB \times CB : BE \times BD$ (Cor. 1. Prop. XVIII. Proportion).

PROP. XX.

35. *In a right-angled triangle* BAC, *if a perpendicular be let fall from the right angle upon the hypothenuſe, it will divide it into two triangles ſimilar to one another and to the whole,* ABD, ADC.

For in the triangles ABD, ABC, the angle B is common to both, and angles D and BAC are right ones; therefore the remaining angles BAD and BCA are equal; therefore the triangles ABD and ABC are ſimilar.

Again, in the triangles ACD and ACB, $\angle C$ is common, $\angle D = CAB$, and therefore $\angle DAC = B$, therefore ACD and ABC are ſimilar; and conſequently ABD and ADC.

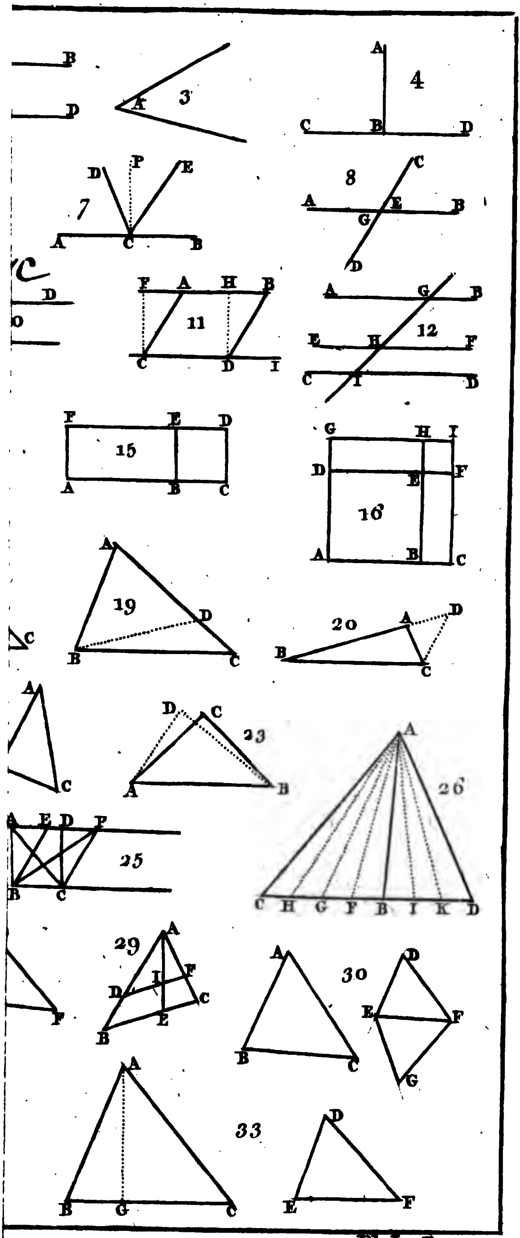
Cor. 1. *The rectangle of the hypothenuſe and either ſegment is equal to the ſquare of the adjoining ſide.*

For $BD : BA :: BA : BC$ (Prop. XIII), and $CD : CA :: CA : CB$ (ibid.); whence $BD \times BC = BA^2$, and $CD \times CB = CA^2$ (Prop. XII. Proportion).

Cor. 2. *The rectangle of the hypothenuſe and perpendicular, is equal to the rectangle of the legs.*

For $BC : AB :: AC : AD$ (Prop. XIII), and $AB \times AC = BC \times AD$ (Prop. XII. Proportion).

Cor.

B
D
A
3
A
4
C B D
D P E
7
A C B
8
C
A E B
G
D
C
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O
F A H B
11
C D I
A G B
12
E H F
C I D
F E D
15
A B C
G H I
D E F
16
A B C
A
19
D
C
B C
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A D
B C
A
C
D C
23
A B
A
26
A E D F
25
B C
C H G F B I K D
F
29
A
I F
D C
B E
A
30
D
E F
B C
G
A
33
D
B G C
E F

Pl. I. *Pa.* 22.

Cor. 3. *The perpendicular is a mean proportional between the ſegments of the hypothenuſe.* FIG. 35.

For $BD : DA : DC$, and $BD \times DC = DA^2$.

Cor. 4. *The ſegments of the hypothenuſe are as the ſquares of the adjoining ſides.*

For by this Prop. $BD : DA :: BA : AC$ (Prop. XIII), and $BD^2 : DA^2 :: BA^2 : AC^2$ (Cor. 3. Prop. XVIII. Proportion). And by Cor. 3. (and Prop. XXIII. Proportion) $BD : DC :: BD^2 : DA^2 :: BA^2 : AC^2$.

Cor. 5. *As the perpendicular, to the hypothenuſe; ſo the rectangle of the ſegments, to the rectangle of the legs.*

For $AD : AB :: CD : CA$, by the ſim. triangles BAD, DAC.

And $BA : BC :: BD : BA$ by the ſim. triangles BAC, BAD.

Therefore $AD : BC :: BDC : BAC$ (Cor. 1. Prop. XVIII. Proportion).

Cor. 6. *The diſtance of the right angle, from the middle of the hypothenuſe, is equal to half the hypothenuſe.*

For let $Bo = oC$, and draw on, or parallel to AC, AB; and draw Ao. Then $Bn = nA$, and $Cr = rA$ (Prop. XII); and the angles at n and r are right (Cor. 1. Prop. IV. I). Then the triangles Bon, Aon, as alſo the triangles Cor, Aor, have two ſides, and the included angle, equal; therefore $Bo = Ao = Co$ (Prop. VI).

PROP. XXI.

In a right-angled triangle BAC, *the ſquare of the hypothenuſe* BC, *is equal to the ſum of the ſquares of the two ſides*, BA, AC. 36.

FIG. 36. For let BG be the ſquare deſcribed on BC, and draw ADF perpendicular to BC, or parallel to CG or BE. Then BA^2 = rectangle of BD and BC or BE (Cor. 1. Prop. XX), = rectangle BF. Alſo the ſquare of AC = rectangle of CD and CB = rectangle CF (ibid.): but rectangle BF + CF = ſquare BG (Ax. 8); therefore BG or the ſquare of $BC = BA^2 + AC^2$.

Cor. 1. *The ſquare of either ſide is equal to the difference between the ſquares of the hypothenuſe and the other ſide*; $BA^2 = BC^2 - AC^2$, *and* CA^2 $BC^2 - BA^2$.

Cor. 2. *The rectangle of the ſum and difference of the hypothenuſe and one of the ſides, is equal to the ſquare of the other ſide.*

For $BA^2 = BC^2 - AC^2$ (Cor. 1) $= \overline{BC + AC} \times \overline{BC - AC}$ (Prop. XII. I).

Cor. 3. *If the ſquare of one ſide of a triangle be equal to the ſum of the ſquares of the other two ſides; then the angle comprehended by them is a right angle.*

For if it was greater or leſs than a right angle, the oppoſite ſide would be greater or leſs than the hypothenuſe of a right-angled triangle (Cor. Prop. VI); and its ſquare greater or leſs than the ſquares of the other ſides.

Cor. 4. *A perpendicular* CA *is the neareſt diſtance of a point* C, *from a right line* BA.

Cor. 5. *In any triangle* ACB, *if a perpendicular be let fall from the oppoſite angle* A, *on the baſe* CB. *The difference of the ſquares of the ſides, is equal to the difference of the ſquares of the ſegments,* $AB^2 - AC^2 = BD^2 - CD^2$.

For $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$ (Cor. 1. Prop. XXI). And $AB^2 - AC^2 = BD^2 - CD^2$ (Ax. 3, 4).

PROP. XXII.

FIG.

In an obtuſe-angled triangle ABC, *if a perpendicular be let fall upon the baſe; or one ſide adjoining to the obtuſe angle* B; *then the ſquare of the ſide oppoſite to that obtuſe angle is equal to the ſum of the ſquares of the two leſſer ſides, together with twice the rectangle of the baſe and the diſtance of the perpendicular from the obtuſe angle:* $AC^2 = AB^2 + CB^2 + 2CBD$. 37.

For $AC^2 = AD^2 + CD^2$ (Prop. XXI) $= AD^2 + CB^2 + BD^2 + 2CBD$ (10. I) $= AB^2 + CB^2 + 2CBD$ (Prop. XXI).

Cor. *The diſtance of the perpendicular from the obtuſe angle,* $BD = \frac{AC^2 - AB^2 - CB^2}{2CB}$.

PROP. XXIII.

If a perpendicular be let fall upon the baſe, or ſide 38.
adjoining to an acute angle B, *of any triangle. Then,* 39.

The ſquare of the ſide oppoſite to that acute angle, together with twice the rectangle, of the baſe, and the diſtance of the perpendicular from the acute angle; is equal to the ſum of the ſquares of the two other ſides: $AC^2 + 2CBD = AB^2 + BC^2$.

For $AC^2 = AD^2 + DC^2$ (Prop. XXI) $= AD^2 + BC^2 + BD^2 - 2BD \times BC$ (Prop. XI, I) $= AB^2 + BC^2 - 2CBD$ (Prop. XXI). And $AC^2 + 2CBD = AB^2 + BC^2$ (Ax. 3).

Cor. *The diſtance of the perpendicular from the acute angle* B *is* $= \frac{AB^2 + BC^2 - AC^2}{2CB}$.

PROP. XXIV.

In any triangle ABC, *let fall a perpendicular* AD 40.
on the baſe BC, *and make* $DF = DB$. *Then* 41.

As the baſe, CB :
to ſum of the ſides, $AC + AB$: :

So

FIG. 40. 41.

$$\left.\begin{array}{l}\textit{So difference of the ſides, } AC - AB : \\ \textit{to difference of the ſegments of the baſe,} \\ \textit{or the alternate baſe}\end{array}\right\} CF.$$

For $CA^2 - AB^2 = CD^2 - DB^2$ (Cor. 5. Prop. XXI); that is,
$\overline{CA + AB} \times \overline{CA - AB} = CF \times CB$ (Prop. XII. I): whence $CB : CA + AB :: CA - AB : CF$ (Cor. 1. Prop. XII. Proportion).

Cor. *The difference of the ſquares of the ſides, is equal to twice the rectangle of the baſe, and the diſtance of the perpendicular from the middle of the baſe,* $CA^2 - AB^2 = 2CB \times oD$.

For if $Co = oB$, then $CA^2 - AB^2 = CF \times CB = \frac{1}{2}CF \times 2CB$; but $\frac{1}{2}CF = Do$; for (Fig. 40) $CF = 2Bo - FB$, and $\frac{1}{2}CF = Bo - BD = Do$. And (Fig. 41) $CF = 2Bo + FB$, and $\frac{1}{2}CF = Bo + BD = Do$.

PROP. XXV.

42. *If an angle* A *of a triangle* BAC *be biſected by a right line* AD, *which cuts the baſe; the ſegments of the baſe will be proportional to the adjoining ſides of the triangle;* $BD : DC :: AB : AC$.

Produce BA, and make $AE = AC$, and draw the line CE; becauſe $AE = AC$, the $\angle ACE = E$ (Prop. III) $= \frac{1}{2}BAC$ (Prop. I) $= BAD$ (hyp.). Therefore DA, CE are parallels (Cor. 3. Prop. IV). Therefore $BA : AE$ or $AC :: BD : DC$ (Prop. XII).

Cor. 1. *If the ſides be as the ſegments of the baſe; the line* AD, *biſects the angle* A.

For ſince $BA : AC$ or $AE :: BD : DC$, DA and CE are parallels (Cor. 1. Prop. XII); and $BAD = \angle E$, and $DAC = ACE = E$ (Prop. III). Whence $BAD = DAC$, and A is biſected by AD.

43. Cor. 2. *If a line biſecting the vertical angle of a triangle cuts the baſe, it will be*

As the ſum of the ſides, BA + AC : FIG.
to their difference, BA — AC : : 43.
So the baſe, BC :
to difference of the ſegments BD — DC.

For BA : AC : : BD : DC (Prop. XXV), and BA + AC : BA — AC : : BD + DC (BC) : BD — DC or 2DO (Prop. XIII. Proportion); where O is the middle point of the baſe BC.

PROP. XXVI.

If an angle A *of a triangle* ABC, *be biſected by a right line* AD, *which cuts the baſe; the ſquare of the biſecting line, together with the rectangle of the ſegments, is equal to the rectangle of the ſides*; $AD^2 + BDC = BAC$. 43.

Produce AD and make the ∠DBP = DAC. Then the three triangles CDA, PDB, and PBA are ſimilar. For AD = PBD = PAB, CDA = PDB (2. I), whence C = P, and ADC = ABP (Cor. 1. Prop. II). Therefore CD : DA : : PD : BD (Prop. XIII), whence DA × PD = CD × BD (12. Proportion). Again, CA : DA : : AP or AD + DP : AB (Prop. XIII), therefore CA × AB = AD^2 + DA × DP (12. Proportion) = AD^2 + CD × BD (Ax. 3).

PROP. XXVII.

In an iſoſceles triangle ABC, *if a line be drawn from the vertex to cut the baſe; the ſquare of that line, together with the rectangle of the ſegments of the baſe, is equal to the ſquare of the ſide*; $BE^2 + AEC = BA^2$. 44.

Let BD be perpendicular to the baſe, then $BA^2 = BD^2 + AD^2$ (Prop. XXI) $= BD^2 + \overline{AE + ED}^2 = BD^2 + AE^2 + ED^2 + 2AED$ (Prop. X. I.) $= BE^2 + AE^2 + 2AED$ (Prop. XXI)

$= BE^2$

FIG. $= BE^2 + AE \times \overline{AE + 2ED}$ (Prop. IX. I)
44. $= BE^2 + AE \times EC$, becauſe $AE + 2ED = EC$. For $2AE + 2ED = AC$, therefore taking away AE, $AE + 2ED = EC$.

PROP. XXVIII.

45. *In any triangle* BAC, *if a line* AD *be drawn from the vertex to the middle of the baſe. The ſum of the ſquares of the ſides, is equal to twice the ſquare of half the baſe, together with twice the ſquare of the line that biſects the baſe*; $AB^2 + AC^2 = 2AD^2 + 2DC^2$.

For $AC^2 + 2CDP = AD^2 + DC^2$ (Prop. XXIII), and $DC = DB$ (hyp.),
therefore $AC^2 = AD^2 + DC^2 - 2CDP$ (Ax. 4);
and $AB^2 = AD^2 + DB^2 + 2CDP$ (Prop. 22),
therefore $AB^2 + AC^2 = 2AD^2 + 2DC^2$ (Ax. 3).

Cor. $AB^2 - AC^2 = (4CDP =)\ 2BC \times DP$.

PROP. XXIX.

46. *If through any point* E, *within a triangle* ABC, *three lines* TQ, VR, PS, *be drawn parallel to the three ſides of the triangle*; *the product or ſolid made by the alternate ſegments of theſe lines, will be equal.* $TE \times PE \times RE = QE \times SE \times VE$.

The triangles TEV, PEQ, SER, and ABC are all ſimilar (7. I), whence

$TE : VE :: AC : BC$ (Prop. XIII).
$PE : QE :: AB : AC$.
$RE : SE :: BC : AB$.

whence $TE \times PE \times RE : VE \times QE \times SE :: AC \times AB \times BC : BC \times AC \times AB$ (Prop. XVIII. Proportion). But the two laſt terms are equal, therefore $TE \times PE \times RE = VE \times QE \times SE$ (Prop. II. Proportion).

PROP. XXX.

FIG. 46.

If three lines AF, BG, CD, *be drawn through any point* E, *within a triangle* ABC, *to the oppoſite ſides; the products of the alternate ſegments of the ſides are equal; that is,* $AG \times CF \times BD = CG \times BF \times AD$.

For drawing TQ, VR, PS parallel to the ſides of the triangle, then

$AG : GC :: TE : QE$ (Cor. Prop. XIII).
$CF : BF :: RE : VE$.
$BD : AD :: PE : SE$.

whence $AG \times CF \times BD : GC \times BF \times AD :: TE \times RE \times PE : QE \times VE \times SE$ (Prop. XVIII. Proportion), but the two laſt are equal (Prop. XXIX); therefore $AG \times CF \times BD = GC \times BF \times AD$ (Prop. II. Proportion).

PROP. XXXI.

Three lines drawn from the three angles of a triangle to the middle of the oppoſite ſides, all meet in one point.

Let BD, AE biſect the oppoſite ſides AC, BC; and through the point of interſection G, draw CGK, and EL, DI parallel to it. 47.

Now ſince $BE = EC$, and $AD = DC$, we have $BL = LK$, and $AI = IK$ (Prop. XII). Alſo ſince $BE = \frac{1}{2}BC$, and $AD = \frac{1}{2}AC$, it will be $EH = \frac{1}{2}CG = DF$ (Prop. XIII). Therefore the triangles DGF, HGE, having all the angles equal (4. I), are ſimilar and equal (Prop. VII); whence $FG = GE$, and conſequently $IK = KL$ (Cor. 2. Prop. XII), therefore $AI = IK = KL = LB = \frac{1}{4}AB$. And $AK = KB$. And therefore if the line CK be drawn through the middle point K, it will paſs through G; otherwiſe the line paſſing through G, would make AK greater or leſſer than KB. This may alſo be demonſtrated from Prop. XXX.

FIG:

47. Cor. *Hence the diſtance of the point of interſection* G, *from any angle, is twice the diſtance from the oppoſite ſide,* $BG = 2GD$, *&c.*

For ſince $BK = 2KI$, and $AK = 2KL$, therefore $BG = 2GD$, and $AG = 2GE$. Alſo ſince $DI = DF + FI = 3HL$ or $3FI$, therefore $2FI = DF = GK = EH = \frac{1}{2}CG$.

PROP. XXXII.

Three perpendicular lines erected on the middle of the three ſides of any triangle, all meet in one point.

48. Let E, F be the middle points of AB, CB; FO, EO two perpendiculars. From O draw OD perpendicular to AC. The right-angled triangles COF, BOF are ſimilar and equal, and $CO = OB$ (Prop. VI); alſo the right-angled triangles BOE, AOE, are ſimilar and equal, whence $BO = OA$ (ibid.); therefore $CO = AO$; therefore in the iſoſceles triangle AOC, the perpendicular OD biſects the baſe AC (Cor. 3. Prop. III): and if it biſects the baſe, it paſſes through O.

Cor. *The point of interſection* O, *of the three perpendiculars, will be equally diſtant from the three angles.*

For the triangles COF, BOF, are ſimilar and equal (Prop. VI), and $OB = OC$. Alſo the triangles COD, AOD, are ſimilar and equal (ibid.), and $CO = AO = BO$.

PROP. XXXIII.

49. *If two right-angled triangles* BID, BED, *be deſcribed upon one hypothenuſe* BD, *lying on different ſides thereof, and the line* EI *drawn to the oppoſite angles; I ſay, the angles* DBI *and* DEI *are equal, which ſtand upon the ſame ſide* DI.

FIG. 49. Make BC = CD; draw ECF and CI. Then CD, CI, CB, and CE are all equal (Cor. 6. Prop. XX). The external angle ICD = CIB + CBI (Prop. I) = 2CBI (Prop. III). Alſo the external angle ICF = EIC + IEC = 2IEC (ibid.). Alſo FCD = CDE + CED = 2CED (ibid.). Therefore by addition ICF + FCD, that is, ICD = 2AED = 2CBI, and AED = CBI, or IED = IBD.

PROP. XXXIV.

Three perpendiculars drawn from the three angles of a triangle, upon the oppoſite ſides, all meet in one point.

50. Let AI, CE be perpendicular to CB, AB; and through the point of interſection D draw BDF; draw CK perpendicular to CA, alſo draw EI.

The oppoſite angles IDC and EDA are equal (2. I), and the angles at E and I are right, therefore the triangles ADE and CDI are ſimilar, whence AD : ED :: CD : DI (Prop. XIII); therefore the triangles ADC, and EDI are ſimilar (Prop. XIV), and angle DEI = DAC = ICK (Prop. XX). But the triangles DBE, DBI are right angled at E and I, whence ∠DEI = DBI (Prop. XXXIII); therefore DBI or FBC = ICK, and therefore BF is parallel to CK (Cor. 3. Prop. IV), or perpendicular to AC. And if BF be perpendicular to AC, it will paſs through D.

PROP. XXXV.

Three lines biſecting the three angles of a triangle, all meet in one point.

51. For let CDF and ADE biſect the angles C, A; and through D, the point of interſection, draw BDG. Then BC : CG :: BD : DG :: BA : AG (Prop. XXV); and BC : BA :: CG : AG (Prop. IV. Proportion), whence BDG biſects the angle B (Cor.

FIG. 51. (Cor. 1. Prop. XXV), therefore the line biſecting the ∠B, paſſes through D.

Cor. 1. *If two lines biſect two angles of a triangle, the point of interſection* D, *is equally diſtant from the three ſides.*

Let D*n*, D*o*, D*p* be perpendicular on the three ſides. Then the triangles BD*n*, BD*o* have one ſide and all the angles equal, therefore D*n* = D*o* (Prop. VII); alſo the triangles AD*o*, AD*p*, have one ſide and all the angles equal; therefore D*o* = D*p* (ibid.) = D*n*.

Cor. 2. *Segment* A*p* + *the oppoſite ſide* BC = *half the ſum of the ſides.*

For half the ſum of the ſides = 2A*p* + 2C*n* + 2B*n*.

PROP. XXXVI.

52. *If the three angles of a triangle be biſected by the lines* AC, BC, DC, *and any one* BC *continued to the oppoſite ſide, and* CP *be drawn perpendicular to that ſide,* AD; *I ſay, the angle* ACE = DCP, *or* ACP = DCE.

For ſince ∠A + B + D = two right angles (Prop. II), therefore CAB + CBA + CDP = a right angle = DCP + CDP (Cor. 2. Prop. II); therefore CAB + CBA or ACE (Prop. I) = DCP.

PROP. XXXVII.

53. *The area of a right-angled triangle* ABC, *is equal to the rectangle under half the perimeter, and its exceſs above the hypothenuſe.*

The perimeter or circumference is the ſum of the three ſides. Now ſince the triangle ABC is right-angled at C, the area = $\frac{AC \times CB}{2}$ (Cor. 2. Prop. X); and $AB^2 = AC^2 + CB^2$ (Prop. XXI), or $AC^2 + CB^2 - AB^2 = 0$. Hence four times the area $= 2AC \times CB = AC^2 + CB^2 + 2ACB - AB^2 = \overline{AC + CB}^2 - AB^2$ (10. I) $= \overline{AC + CB + AB} \times \overline{AC + CB - AB}$ (12. I). FIG. 53.

And the area $= \frac{AC+CB+AB}{2} \times \frac{AC+CB-AB}{2}$.

But $\frac{AC+CB-AB}{2} = \frac{AC+CB+AB}{2} - AB$.

Cor. *The area of a right-angled triangle, is equal to the rectangle under the two exceſſes, of half the perimeter above each ſide;* $\frac{AC+CB+AB}{2} - BC$, *and* $\frac{AC+CB+AB}{2} - AC$.

For $\frac{AC+CB+AB}{2} - CB = \frac{AB+AC-BC}{2}$, and $\frac{AC+CB+AB}{2} - AC = \frac{AB+CB-AC}{2}$, and $\frac{AB+AC-BC}{2} \times \frac{AB+BC-AC}{2} = \frac{AB+\overline{AC-CB}}{2} \times \frac{AB-\overline{AC-CB}}{2} = \frac{AB^2-\overline{AC-CB}^2}{4}$ $= \frac{AB^2 - AC^2 - CB^2 + 2ACB}{4}$ (Prop. XXI) $= \frac{ACB}{2}$ = area (Cor. 2. Prop. X).

PROP. XXXVIII.

54. *In any triangle* ABC; *add the three ſides together into one ſum; and likewiſe from the ſum of every two ſides, ſubtract the third; and you will have three remainders. Then take the product of the ſaid ſum, and one of the remainders; and likewiſe the product of the other two remainders.*

Then I ſay, four times the area of the triangle, is a mean proportional, between theſe two products.

Take AE, and AF, equal to AC, and draw CF, CE; alſo draw CD perpendicular to AB. Then

FIG. 54. $AB \times CD =$ twice the area (Cor. 2. Prop. X). And the angle FCE is a right angle; for $AFC = ACF$ (Prop. III), and $AEC = ACE$ (ibid.); therefore $AFC + AEC = ACF + ACE = FCE$ (Ax. 3) $=$ a right angle (Cor. 2. Prop. II). And $AD = \frac{AB^2 + AC^2 - CB^2}{2AB}$ (Cor. Prop. XXIII).

Now $DE = AE - AD = AC - AD$ $= \frac{AC \times 2AB - AB^2 - AC^2 + CB^2}{2AB} = \frac{\overline{CB + AB - AC} \times \overline{CB + AC - AB}}{2AB}$ (11. I).

Alſo $FD = FE - DE = 2AC - DE = \frac{2AC \times 2AB}{2AB} - DE = \frac{2AC \times 2AB - AC \times 2AB + AB^2 + AC^2 - CB^2}{2AB}$ $= \frac{AB^2 + AC^2 + 2AC \times AB - CB^2}{2AB} = \frac{\overline{AB + AC + BC} \times \overline{AB + AC - BC}}{2AB}$ (12. I); but DC is a mean between DE and DF (Cor. 3. Prop. XX), therefore $DC \times 2AB$ is a mean between $DE \times 2AB$ and $DF \times 2AB$ (Prop. V. Proportion); that is, four times the area of the triangle ABC, is a mean proportional, between $\overline{CB + AB - AC} \times \overline{CB + AC - AB}$, and $\overline{AB + AC + BC} \times \overline{AB + AC - BC}$.

55. Cor. 1. *From half the ſum of the three ſides of any triangle* ABC, *ſubtract each ſide ſeparately. Then take the product of that half ſum and one remainder; and alſo the product of the other two remainders.*

Then I ſay, the area of the triangle is a mean proportional between theſe two products.

For $\frac{CB + AB - AC}{2} \times \frac{CB + AC - AB}{2}$: area ABC : $\frac{AB + AC + BC}{2} \times \frac{AB + AC - BC}{2}$

(Cor. 1. Prop. V. Proportion) are in continual proportion (Prop. XXXVIII). FIG. 55.

But $\frac{CB + AB - AC}{2} = \frac{CB + AB + AC}{2} - AC$.

and $\frac{CB + AC - AB}{2} = \frac{CB + AB + AC}{2} - AB$.

and $\frac{AB + AC - BC}{2} = \frac{CB + AB + AC}{2} - BC$.

therefore, &c.

Cor. 2. *Let* $S = AC + BC$, $D = AC - BC$, *then the area* ABC *is a mean proportional between* $\frac{1}{4} \times \overline{SS - AB^2}$, *and* $\frac{1}{4} \times \overline{AB^2 - DD}$. 56.

For $\frac{1}{4} \times \overline{SS - AB^2} = \frac{S + AB}{2} \times \frac{S - AB}{2} = \frac{AC + BC + AB}{2} \times \frac{AC + BC - AB}{2}$, and $\frac{1}{4} \times \overline{AB^2 - DD} = \frac{AB + D}{2} \times \frac{AB - D}{2} = \frac{AB + AC - BC}{2} \times \frac{AB + BC - AC}{2}$, which is the ſame, as Cor. 1. ſuppoſing two terms in the extremes to change places, by Cor. 3. Prop. XII. Proportion.

PROP. XXXIX.

The ſquare of the ſide of an equilateral triangle, is to the area; as 4 *to* $\sqrt{3}$.

Let CD be perpendicular to AB, then $AD = DB = \frac{1}{2}AB$. Then $CD^2 = CA^2 - AD^2$ (Cor. 1. Prop. XXI) $= AB^2 - \frac{1}{4}AB^2 = \frac{3}{4}AB^2$. And $CD = \sqrt{\frac{3AB^2}{4}} = \frac{AB}{2}\sqrt{3}$. But the area of the triangle is $AB \times \frac{1}{2}CD = AB \times \frac{AB}{4}\sqrt{3}$, and $4 \times \text{area} = AB^2 \times \sqrt{3}$ (Cor. 2. Prop. X); whence $AB^2 : \text{area} :: 4 : \sqrt{3}$. 57.

Cor. *The ſquare of the perpendicular is equal to* $\frac{3}{4}$ *the ſquare of the ſide*; $CD^2 = \frac{3}{4}CA^2$.

For $CD^2 = CA^2 - AD^2$ (Cor. XXI) $= CA^2 - \frac{1}{4}CA^2 = \frac{3}{4}CA^2$.

BOOK III.

Of Quadrangles and Polygons.

DEFINITIONS.

FIG. 1. A *Quadrangle or quadrilateral*, is a plane figure bounded by four right lines.

58. 2. A *parallelogram* is a quadrangle whoſe oppoſite ſides are parallel, as AGBH. The line AB drawn to the oppoſite corners is called the *diameter* or *diagonal*. And if two lines be drawn parallel to the two ſides, through any point of the diagonal; they divide it into ſeveral others, and then C, D are called *parallelograms about the diameter:* and E, F *the complements:* and the figure EDF a *gnomon.*

3. A *rectangle* is a parallelogram whoſe ſides are perpendicular to one another.

4. A *ſquare* is a rectangle of four equal ſides.

59. 5. A *rhombus* is a parallelogram, whoſe ſides are equal, and angles oblique.

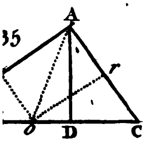
58. 6. A *rhomboides* is a parallelogram, whoſe ſides are unequal, and angles oblique.

60. 7. A *trapezoid* is a quadrangle, having only two ſides parallel.

61. 8. A *trapezium* is a quadrangle, that has no two ſides parallel.

62. 9. A *polygon* is a plane figure encloſed by many right lines. If all the ſides and angles are equal, it is called a *regular polygon,* and denominated according to the number of ſides, as a *pentagon* 5 ſides, a *hexagon* 6, a *heptagon* 7, &c.

10. The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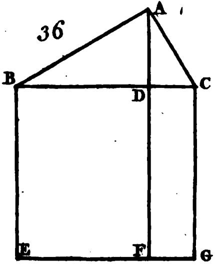
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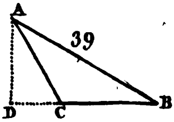
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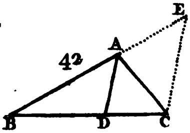
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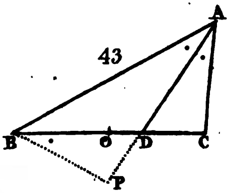
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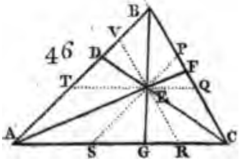
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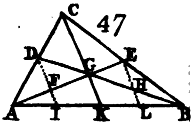
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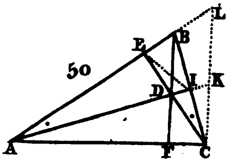
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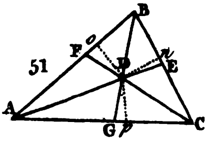
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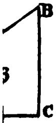

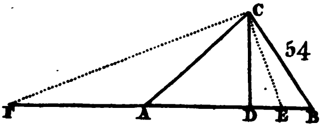
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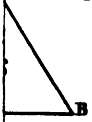

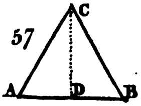
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10. The *diagonal* of a quadrangle or polygon, is a line drawn between any two oppoſite corners of the figure, as AB. FIG. 62.

11. The *hight* of a figure is a line drawn from the top, perpendicular to the *baſe*, or oppoſite ſide, on which it ſtands.

12. *Like or ſimilar figures*, are thoſe whoſe ſeveral angles are equal to one another; and the ſides about the equal angles, proportional.

13. *Homologous ſides* of two figures, are thoſe between two angles, reſpectively equal.

14. The *perimeter* or *circumference* of a figure, is the compaſs of it, or ſum of all the lines that incloſe it.

15. The *internal angles* of a figure, are thoſe on the inſide, made by thoſe lines that bound the figure, ADC. 76.

16. The *external angle* of a figure, is the angle made by one ſide of a figure, and the adjoining ſide drawn out, BAF. 77.

PROP. I.

In any parallelogram the oppoſite ſides, and angles, are equal; and the diagonal divides it into two equal triangles: $AB = CD$, $AC = BD$, *and triangle* $ABD = ADC$, *&c.* 63.

For ſince AB, and CD are parallel (Def. 2), $\angle BAD = ADC$ (4. I): alſo, becauſe AC and BD are parallel, $BDA = CAD$ (ibid.). Therefore the triangles ABD and DCA, are equal in all reſpects (7. II).

PROP. II.

The diagonals of a parallelogram, interſect each other in the middle.

In the triangles APC, BPD, $\angle CAP = BDP$, and $ACP = DBP$ (4. I), and $\angle BPD = APC$ (2. I), and $AC = BD$ (Prop. I); therefore $AP = PD$, and $CP = PB$ (7. II). 64.

FIG.

PROP. III.

Any line BC *paſſing through the middle of the diagonal of a parallelogram* P, *divides the area into two equal parts.*

65. For in the triangles ABP, and DCP; AP = PD (Prop. II); and all the angles are equal (4. I). Therefore the triangle ABP = DCP (7. II); and BP = PC (ibid.). And ſince triangle AED = AFD (Prop. I); the remainders BPDE and CPAF are equal; therefore BPDE + PDC = CPAF + APB, that is, EBCD = BAFC.

Cor. *Any right line* BC *drawn through the middle point* P *of the diagonal of a parallelogram, is biſected in that point*; BP = PC.

PROP. IV.

66. *In any parallelogram* ABDC, *the compliments* CI, *and* IB, *are equal.*

For triangle ADC = ABD (Prop. I), and AHI = AGI, and IED = IFD (ibid.); therefore parallelogram HE = parallelogram GF (Ax. 4).

PROP. V.

66. *The parallelograms* HG, EF, *which are about the diameter* AD, *of any parallelogram* CB, *are ſimilar to the whole* CB, *and to one another.*

The parallelograms HG, EF are equiangular to the whole CB (4. I), and to one another. The triangles ACD, ABD, as alſo AHI, AGI, and IED, IFD, are ſimilar and equal (Prop. I). Therefore AH : HI or AG : : AC : CD or AB : : IE : ED or IF, therefore the parallelograms are like (Def. 12).

PROP.

PROP. VI.

FIG. 67. *Parallelograms* ABCD, *and* EBCF, *ſtanding upon the ſame baſe and between the ſame parallels, are equal.*

For AD = BC = EF (Prop. I); add DE, then AE = DF, and AB = DC (Prop. I), and ∠A = CDF (Cor. 1. 4. I). Therefore triangle ABE = DCF (6. II); ſubtract DGE; then the figure ABGD = EGCF; add BGC, then ABCD = BEFC.

Cor. 1. *Parallelograms of equal baſes and hights, are equal.*

For if their baſes be laid upon one another, the tops of both will fall in the ſame parallel, being of equal hight; and therefore they are equal (this prop.).

Cor. 2. *Every parallelogram is equal to the rectangle of its baſe and hight.*

Cor. 3. *Figures of the ſame area, may have their compaſs vaſtly different. And figures of equal compaſs may contain very different areas.*

PROP. VII.

A parallelogram is double to a triangle of the ſame or an equal baſe and hight.

For the triangle ACD = ABD (Prop. I), that is, 63. the triangle ACD, on the baſe CD, is half the parallelogram ACDB on the ſame baſe CD, and between the ſame parallels. And ſince any triangle of an equal baſe and hight is = ACD, and any parallelogram of the ſame or an equal baſe and hight = ACDB. Therefore any triangle is half the parallelogram of the ſame or equal baſes and hights.

FIG.

PROP. VIII.

68. *Parallelograms of the ſame hight, are to one another as their baſes;* DC : GF : : BC : GH.

Draw the diameters BA, EH. Then the triangles BCA, GHE, of the ſame hight, are as their baſes BC, GH (11. II). Therefore 2BCA : 2GHE : : BC : GH (Prop. V. Proportion): that is, parallelogram BCAD : parallelogram GHFE : : baſe BC : baſe GH.

Cor. 1. *Parallelograms of equal baſes, are as their hights.*

By Cor. 2. Prop. VI. as likewiſe

Cor. 2. *Parallelograms are to one another, as their baſes and hights.*

PROP. IX.

69. *Equal parallelograms having one angle equal to one; have the ſides about the equal angles reciprocally proportional. If* ABCD = EFGH, *then* AB : BG : : BE : BC.

Let the oppoſite angles at B be equal; produce DC and FG to H. Then AB : BG : : BD : BH (Prop. VIII) : : BF : BH (Ax. 6. Proportion) : : BE : BC (Prop. VIII).

Cor. 1. *Thoſe parallelograms are equal; which have one angle equal to one; and the ſides about the equal angles, reciprocally proportional.*

For BD : BH : : AB : BG (Prop. VIII) : : BE : BC (hyp.) : : BF : BH (Prop. VIII). Therefore parallelogram BD = parallelogram BF.

Cor. 2. *Equal parallelograms, have their baſes and hights, reciprocally proportional.*

Cor. 3. *If four lines are proportional; the rectangle of the means, is equal to the rectangle of the extremes.*

PROP.

PROP. X.

FIG. 70. *Equiangular parallelograms* AC, EG, *are in the complicate ratio of their homologous ſides,* ABC, EBG.

Produce DC, FG to H. Then parallelogram AC : BH : : AB : BG (Prop. VIII), and parallelogram BH : BF : : CB : BE (ibid.). Therefore parallelogram AC : parallelogram BF : : AB × CB : BG × BE (Cor. 1. Prop. XVIII. Proportion).

Cor. 1. *Parallelograms are to one another, in the complicate ratio of their baſes and hights.*

Cor. 2. *The rectangle of two lines, is a mean proportional between their ſquares.*

For ſuppoſing AC, EG, to be ſquares; then AC : BH : : (AB : BG : : BC : BE : :) BH : BF.

PROP. XI.

71. *In any parallelogram* AD, *the ſum of the ſquares of the diagonals, is equal to the ſum of the ſquares of all the ſides:* $AD^2 + CB^2 = CA^2 + AB^2 + BD^2 + DC^2$.

For CE = EB, and AE = ED (Prop. II). Alſo $CD^2 + DB^2 = 2DE^2 + 2CE^2$ (28. II). And $2CD^2 + 2DB^2 = 4DE^2 + 4CE^2$, that is, $CD^2 + AB^2 + DB^2 + CA^2 = DA^2 + CB^2$.

PROP. XII.

72. *If from any point* O, *in the rectangle* AD, *lines be drawn to all the angles; the ſum of the ſquares of the lines drawn to the oppoſite corners, will be equal:* $AO^2 + OD^2 = BO^2 + OC^2$.

Draw AD, BC, to interſect in P, then AD = CB (6. II), and their halfs, AP = PC = PD. Then $CO^2 + OB^2 = 2CP^2 + 2OP^2$ (28. II) $= 2AP^2 + 2OP^2 = AO^2 + OD^2$ (28. II).

PROP.

PROP. XIII.

FIG. 73. *In any trapezium* ABDC, *let* E, F *be the middle points of the diagonals*, AD, BC. *Then the ſum of the ſquares of the ſides, is equal to the ſum of the ſquares of the diagonals, together with four times the ſquare of the diſtance, between the middle points of the diagonals:* $AB^2 + BD^2 + CD^2 + CA^2 = AD^2 + CB^2 + 4EF^2$.

For $AE^2 + ED^2 = 2AF^2 + 2EF^2$ (28. II). Alſo $AB^2 + AC^2 = 2CE^2 + 2AE^2$ (ibid.); alſo $BD^2 + DC^2 = 2CE^2 + 2DE^2$. And adding the two laſt equations, $AB^2 + BD^2 + DC^2 + CA^2 = 4CE^2 + 2AE^2 + 2ED^2 = CB^2 + 4AF^2 + 4EF^2 = CB^2 + AD^2 + 4EF^2$.

PROP. XIV.

74. *In any trapezium* ADBC, *let* E, F, *be the middle points of two oppoſite ſides. Then the ſum of the ſquares of the other two ſides, together with the ſquares of the diagonals, is equal to the ſum of the ſquares of the biſected ſides, together with four times the ſquare of the diſtance of theſe middle points:* $AC^2 + DB^2 + AB^2 + CD^2 = AD^2 + CB^2 + 4EF^2$.

Draw AE, ED. Then $AE^2 + ED^2 = 2AF^2 + 2EF^2$ (28. II); and $AB^2 + AC^2 = 2CE^2 + 2AE^2$ (ibid.), and $DB^2 + DC^2 = 2CE^2 + 2DE^2$ (ibid.). Add the two laſt equations, $AB^2 + AC^2 + DB^2 + DC^2 = 4CE^2 + 2AE^2 + 2ED^2 = CB^2 + 4AF^2 + 4EF^2 = CB^2 + AD^2 + 4EF^2$.

PROP. XV.

75. *In any trapezium* ADBC, *if lines be drawn to the middle of the oppoſite ſides; the ſum of the ſquares of the diagonals, is equal to twice the ſum of the ſquares of the biſecting lines:* $AB^2 + CD^2 = 2EF^2 + 2PQ^2$.

For $AB^2 + DC^2 + BD^2 + CA^2 = AD^2 + CB^2 + 4EF^2$ (Prop. XIV). FIG. 75.

And $AB^2 + DC^2 + BC^2 + DA^2 = AC^2 + DB^2 + 4PQ^2$ (ibid.).

and adding theſe equations,
$2AB^2 + 2DC^2 + BD^2 + CA^2 + BC^2 + DA^2 = AD^2 + CB^2 + AC^2 + DB^2 + 4EF^2 + 4PQ^2$, and ſubtracting what is common, $2AB^2 + 2DC^2 = 4EF^2 + 4PQ^2$, and $AB^2 + DC^2 = 2EF^2 + 2PQ^2$.

PROP. XVI.

The ſum of the four internal angles of any quadrilateral figure, is equal to four right angles.

Draw the diagonal AC; then the ſum of all the angles in the triangle ABC, or ADC, is two right angles (2. II); therefore the ſum of both is four right angles. 76.

Cor *If two angles of a quadrangle be right angles; the ſum of the other two amounts to two right angles.*

PROP. XVII.

The ſum of all the internal angles of a polygon, makes twice as many right angles, abating four, as the polygon has ſides.

For drawing lines from all the angles, to a point O within the figure, it comes to be divided into as many triangles, as the figure has ſides or angles. And each triangle contains two right angles (2. II), ſo theſe amount to twice as many right angles, as the figure has ſides; but the angles at O are to be abated, and theſe amount to four right angles (Cor. 1. Prop. 1. I). 77.

Cor. *Hence all right-lined figures, of the ſame number of ſides, have the ſum of all the internal angles equal.*

PROP.

FIG.

PROP. XVIII.

The ſum of the external angles of any polygon, is equal to four right angles.

77. For all the internal angles, together with the external angles at the points A, B, C, *&c.* make twice as many right angles, as the figure has ſides (1. I); and the ſum of all the angles of the triangles ABO, BCO, *&c.* amounts to the ſame (2. II). Take away all the angles, EAB, ABC, *&c.* and there remains all the external angles A, B, C, *&c.* equal to all the angles at O, that is, four right angles (Cor. 1. (Prop. 1. I).

Cor. *All right-lined figures, have the ſum of their external angles equal.*

SCHOLIUM.

78. If any of the angles be greater than two right angles, as A; the external angle will run into the figure, and muſt be ſubtracted from the ſum of the reſt.

PROP. XIX.

79. *In two ſimilar figures* AC, PR; *if two lines* BE, QT, *be drawn after a like manner, as ſuppoſe, to make the angle* CBE = RQT; *then theſe lines have the ſame proportion, as any two homologous ſides of the figure,* BC *to* QR, *&c.*

Since $\angle$CBE = RQT, and R = C (hyp.); therefore BE : QT : : BC : QR (13. II) : : BA : QP (Def. 12) : : AD : PS (ibid.) : : DC : SR. Alſo BC : CE : : QR : RT; and BC : BE : : QR : QT, *&c.*

Cor. 1. *Hence all ſimilar figures are made up of ſimilar triangles.*

Draw

FIG. 79. Draw BD, QS; and AC, PR; then BE : QT :: BC : QR (this prop.) :: CD : RS (Def. 12) :: CE : RT (this prop.) :: DE : ST (Prop. VIII. Proportion); therefore the triangles BCE and QRT are ſimilar; and BED and QTS are ſimilar.

Again, the $\angle A = P$, and AB : AD :: PQ : PS (Def. 12); therefore BAD, QPS are ſimilar (14. II). Alſo $\angle B = Q$, and AB : BC :: PQ : QR, therefore ABC and PQR are ſimilar (14. II). Laſtly, $\angle D = S$, and AD : DC :: PS : SR (Def. 12); therefore ADC, PSR are ſimilar (14. II).

Cor. 2. *Hence it may be laid down, as a diſtinguiſhing property of ſimilar figures, that they are made up of ſimilar triangles, placed in the ſame order.*

PROP. XX.

All ſimilar figures are to one another as the ſquares of their homologous ſides.

80. Let AD, PS be ſimilar polygons; draw AC, AD, PR, PS, which will divide the figures into triangles (Cor. 1. Prop. XIX).

Becauſe AB : PQ :: AC : PR :: AD : PS (13. II); therefore

$AB^2 : PQ^2$ :: triangle ABC : PQR (18. II).
and $AB^2 : PQ^2 :: AC^2 : PR^2$ :: triangle ACD : PRS (ibid.).
and $AB^2 : PQ^2 :: AD^2 : PS^2$ :: triangle ADE : PST (ibid.).
therefore $AB^2 : PQ^2$ :: triangle ABC + ACD + ADE : triangle PQR + PRS + PST (Prop. X. Proportion) :: figure ABCDE : figure PQRST.

Cor. *If three lines* A, B, C *be in continual proportion; then as the firſt to the third, ſo any figure deſcribed on the firſt, to a ſimilar one upon the ſecond.*

For A : C :: $A^2 : B^2$ (Prop. XXIII. Proportion) :: figure upon A : figure upon B (this prop.).

PROP.

FIG.

PROP. XXI.

81. *If four lines be proportional,* AB : DE :: GH : LM; *ſimilar figures, alike deſcribed upon, two and two, ſhall alſo be proportional:* ABC : DEF :: GHIK : LMNO.

And if four figures be proportional, and two and two be ſimilar; their like ſides ſhall be proportional.

For ſince AB : DE :: GH : LM (hyp.),
therefore $AB^2 : DE^2 :: GH^2 : LM^2$ (Cor. 3. (Prop. XVIII. Proportion).
whence ABC : DEF : GHI : LMN (Prop. XX).
Again, if the figures be ſimilar,
and ABC : DEF :: GHIK : LMNO (hyp.).
then $AB^2 : DE^2 :: GH^2 : LM^2$ (Prop. XX).
whence AB : DE :: GH : LM (Cor. 3. 18. Proportion).

PROP. XXII.

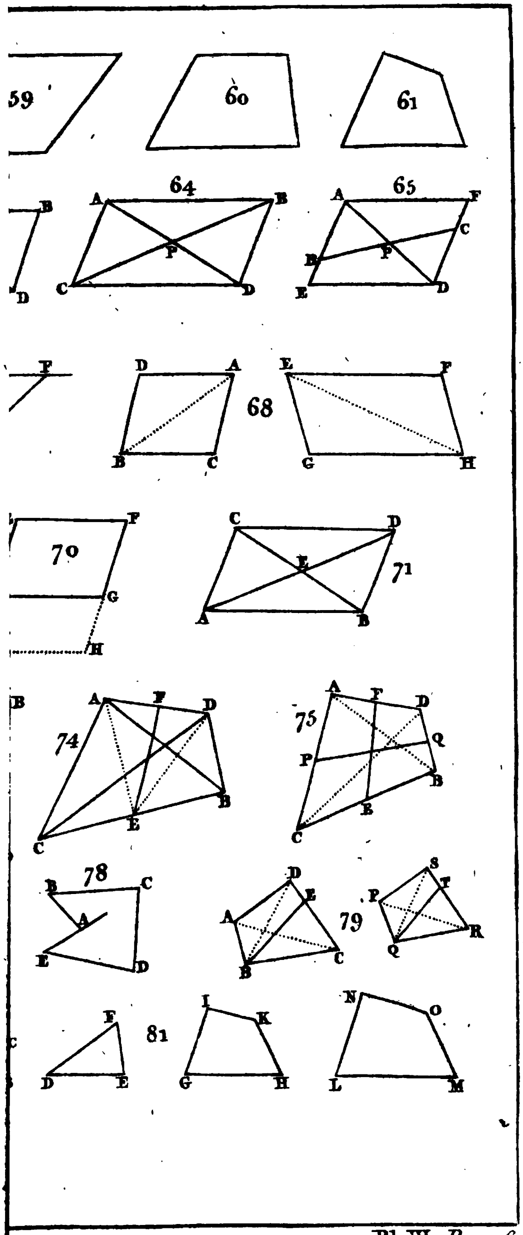
82. *Any figure deſcribed on the hypothenuſe of a right-angled triangle, is equal to two ſimilar figures deſcribed the ſame way upon the two ſides:* BFC = ALC + AGB.

For fig. $BCF : \begin{matrix} CAL \\ BAG \end{matrix} :: BC^2 : \begin{matrix} CA^2 \\ AB^2 \end{matrix}$ (Prop. XX).
therefore, $BCF : CAL + BAG :: BC^2 : CA^2 + AB^2$ (14. Proportion).
But $BC^2 = CA^2 + AB^2$ (21. II); therefore BCF = CAL + BAG (Prop. II. Proportion).

PROP. XXIII.

83. *The area of a trapezoid* ABCD, *is equal to the rectangle of half the ſum of the parallel ſides, and the perpendicular between them:* $\frac{BA + CD}{2} \times BP$.

Draw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Pl. III. *Pa. 46.*

Draw AF parallel to BD, and BP perpendicular to CD. Then the area ABDF $= AB \times BP$ or $FD \times BP$ (Cor. 2. Prop. VI) $= \frac{AB + FD}{2} \times BP$. And the area of the triangle ACF $= \frac{1}{2} CF \times BP$ (Cor. 2. Prop. X. II). Therefore CAF + AFDB or the trapezoid CABD $= \frac{AB + DF + CF}{2} \times BP = \frac{AB + CD}{2} \times BP$. FIG. 83.

PROP. XXIV.

The area of a trapezium ABDF, *is equal to half the rectangle under the diagonal* AD, *and the ſum of the perpendiculars falling thereon from the oppoſite angles:* $AD \times \frac{BC + EF}{2}$. 84.

For the triangle ABD $= \frac{AD \times BC}{2}$ (Cor. 2. 10. II); and the triangle AFD $= \frac{AD \times FE}{2}$ (ibid.): therefore ABD + AFD or the trapezium ABDF $= \frac{AD \times \overline{BC + FE}}{2}$.

PROP. XXV.

Any regular figure ABCDE, *is equal to a triangle, whoſe baſe is the perimeter* ABCDEA; *and hight, the perpendicular* OP, *drawn from the center, perpendicular to one ſide.* 85.

Two perpendiculars, as PO, ſtanding on the middle of two ſides, meet in the center, O (9. II). Or two angles A, B biſected by two right lines, meet in the center, O (Cor. 1. 3. II): whence all the lines OA, OB, OC are equal; and all perpendiculars drawn from O, upon AB, BC, CD, *&c.* are equal. And all the triangles AOB, BOC, *&c.* are

FIG. 85. equal and ſimilar. The ſum of all the triangles make up the figure, that is, $\frac{AB \times OP}{2}$, $\frac{BC \times OP}{2}$, *&c.* or $\frac{AB + BC + CD + DE + EA}{2} \times OP =$ figure, or a triangle whoſe baſe is ABCDEA, and hight OP = the figure.

Cor. *The area of a regular polygon is equal to the rectangle of one ſide into the perpendicular from the center upon that ſide, and that multiplied by half the number of ſides.*

SCHOLIUM.

Any polygon, regular or irregular, may be divided in as many triangles, leſs 2, as the figure has ſides; by drawing diagonal lines.

PROP. XXVI.

Only three ſorts of regular figures can fill up a plane ſurface; and theſe are ſix triangles, four ſquares, and three hexagons.

It is required to place ſome number of theſe figures, with their angles upon one point, ſo that being joined cloſe together, they may fill the whole ſpace around it, and leave no vacancy.

It is plain the angles about one point are four right angles (Cor. 1. 1. I), which want to be filled up. Now if the angles of the ſeveral figures be computed by Prop. XVII, they will be found as follows.

86. A triangle $\frac{2}{3}$ of a right angle = A.
A ſquare 1 right angle = B.
A pentagon $1\frac{1}{5}$ right angle.
A hexagon $1\frac{1}{3}$ a right angle = C.
&c.

Now $\frac{2}{3}$ of a right angle 6 times repeated, makes 4 right angles, and therefore fills all the ſpace; that is, 6 angles of an equilateral triangle fills it.

Alſo 4 angles of a ſquare (or 4 × 1), makes 4 right angles. FIG. 86.

But 3 angles of a pentagon (or $3 \times 1\frac{1}{5}$) falls ſhort; and 4 angles (or $4 \times 1\frac{1}{5}$) exceeds.

Alſo 3 angles of a hexagon (or $3 \times 1\frac{1}{3}$) makes 4 right angles. And theſe are all; for

The angle of a heptagon (and other figures) is bigger, and therefore 3 angles will exceed 4 right ones. And to have two angles, each muſt be right angles, which is abſurd.

BOOK IV.

Of the Circle, and inſcribed and circumſcribed Figures.

DEFINITIONS.

FIG. 87. 1. A *Circle* is a plane figure deſcribed by a right line moving about a fixt point, ABD. Or it is a figure bounded by one line equidiſtant from a fixt point.

2. The *center* of a circle, is the fixt point about which the line moves, C.

3. The *radius*, is the line that deſcribes the circle, CA.

Cor. *All the radii of a circle, are equal.*

4. The *circumference* is the line deſcribed by the extreme end of the moving line, ABDA.

88. 5. The *diameter*, is a line drawn through the center, from one ſide to the other, AD.

6. A *ſemicircle*, is half the circle, cut off by the diameter, as ABD.

7. A *quadrant*, or quarter, is the part between two radii perpendicular to one another, as CDE.

89. 8. An *arch* is any part of the circumference, AB.

9. A *ſector*, is a part bounded by two radii, and the arch between them, ACB.

10. A *ſegment*, a part cut off by a right line, DEF, or DABF.

11. A *cord*, a right line drawn through the circle, as DF. FIG. 89.

12. *Angle at the center*, is that whoſe angular point is at the center ACB.

13. *Angle at the circumference*, is when the angular point is in the circumference, BAD. 90.

14. *Angle in a ſegment*, is the angle made by two lines drawn from ſome point of the arch of that ſegment, to the ends of the baſe; as BCD is an angle in the ſegment BCD.

15. *Angle upon a ſegment*, is the angle made in the oppoſite ſegment, whoſe ſides ſtand upon the baſe of the firſt; as BAD, which ſtands upon the ſegment BCD.

16. A *tangent* is a line touching a circle, which produced, does not cut it, as GAF.

17. Circles are ſaid *to touch* one another, which meet, but do not cut one another.

18. *Similar arches*, or *ſimilar ſectors*, are thoſe bounded by radii that make the ſame angle.

19. *Similar ſegments* are thoſe which contain ſimilar triangles, alike placed.

20. A *figure* is ſaid to be *inſcribed in a circle*, or a *circle circumſcribed about a figure*; when all the angular points of the figure are in the circumference of the circle.

21. A *circle* is ſaid to be *inſcribed* in a figure, or a *figure circumſcribed about* a circle; when the circle touches all the ſides of the figure.

22. One *figure* is *inſcribed* in another, when all the angles of the inſcribed figure, are in the ſides of the other.

PROP. I.

The cord of any arch AB, *falls intirely within the circle.* 91.

For draw CA, CB; and CD to any point of the cord; then $\angle A = B$ (3. II). And $\angle CDB =$

FIG. 91. $A + ACD$ (1. II) $= B + ACD$, therefore CDB is greater than B, conſequently CB is greater than CD, (4. II); therefore D is within the circle.

PROP. II.

92. *The radius* CR, *biſects any cord at right angles, which paſſes not through the center, as* AB.

For draw AC, BC, and if $AF = FB$, then ſince $AC = CB$, and CF common; therefore $CFA = CFB$ (8. II) $=$ a right angle; and angle $ACF = BCF$.

Or if $AFC = CFB$ and $A = B$, then $ACF = BCF$; and CF being common, $AF = FB$. This prop. follows from Cor. 3. Prop. III. Book II.

Cor. 1. *If a line biſects a cord at right angles, it paſſes through the center of the circle.*

Cor. 2. *The radius that biſects the cord, alſo biſects the arch.*

For ſince $ACR = RCB$. If CBR be laid upon CAR, the point B will fall upon A, and therefore $RB = RA$.

Cor. 3. *If two right lines do not both paſs through the center, they cannot both be biſected by each other.*

For if they could, they muſt both make right angles with the radius.

PROP. III.

93. *In a circle, equal cords* AB, GD, *are equally diſtant from the center,* C.

For let CE, CF be perp. to the cords; and draw CD, CA; then in the triangles, ACF, DCE, $AC = CD$, $AF = DE$ being half the cords (Prop. II); and angles at F, E right; and the angles at C, both acute, therefore $CF = CE$ (9. II).

Cor. *If ſeveral lines be drawn through a circle, the greateſt is the diameter, and thoſe that are nearer the center* HI, *are greater than thoſe that are farther off*, DG. FIG. 93.

For draw CH, then CH is greater than OH (4. II), and therefore 2CH or the diameter is greater than HI. And ſince ∠ HCI is greater than DCG, HI is greater than DG (Cor. 6. II).

PROP. IV.

If from a point G, *out of the center, ſeveral lines* GD, GE, *&c. be drawn, the greateſt is that* GF *which paſſes through the center, and thoſe nearer to* GF *are greater than thoſe further off.* 94.

Alſo GH *(the remainder to* GF) *is the leaſt, and thoſe nearer to it, as* GA, *are leſs than thoſe further off*, GB.

Draw CE, CD, CA, CB, from the center C. Then GC + CE or GF is greater than GE (5. II). Alſo in the triangles GCE, GCD; GC, EC are equal to GC, DC; but ∠ ECG is greater than DCG; therefore EG is greater than DG.

Alſo CG + GD is greater than CD or CH, take away CG, and GD is greater than GH. After the ſame manner GA is greater than GH; and GB greater than GA.

Cor. 1. *Only two lines drawn from* G *to the circumference can be equal; and lie on different ſides of the diameter* HF.

For no two lines on the ſame ſide can be equal.

Cor. 2. *If from any point, three equal right lines can be drawn to the circumference; that point is the center*, C.

Cor. 3. *No circle can cut another in more than two points.*

FIG. 94. For then three equal lines might be drawn from a point out of the center to the circumference; which is abſurd.

PROP. V.

95. *If from a point* G *without a circle, ſeveral right lines be drawn to cut it. Of thoſe that paſs to the concave part, the greateſt is that* GF *which paſſes through the center, and thoſe nearer to* GF *are greater than thoſe further off.*

But of thoſe that go to the convex part, the leaſt is that GH, *which continued would paſs through the center, and thoſe nearer to that, as* GA, *are leſs than thoſe further off,* GD.

For in the triangle GCE, GC + CE or GF is greater than GE. And in the triangles GCE, GCB; GC, CE are equal to GC, CB, and ∠GCE greater than GCB, therefore GE is greater than GB.

Alſo in the triangle CGA, CA + AG is greater than CG or CH + HG (5. II); take away CA = CH, and AG is greater than HG. And in the triangles CAG, CDG; CG, CA are equal to CG, CD; and angle GCA leſs than GCD; therefore GA is leſs than GD (Cor. 6. II).

Cor. 1. *There can only two equal lines be drawn from the point* G *to the circumference of the circle.*
For no two are equal on one ſide of GF.

Cor. 2. *The greateſt to the convex part, or the leaſt to the concave part, is the tangent to the circle.*

PROP. VI.

96. *In any circle, if ſeveral radii be drawn making equal angles; the arches and ſectors comprehended thereby will be equal, if* ACB = BCD; *then, arch* AB = *arch* BD; *and ſector* ACB = BCD.

For

FIG. 96. For ſince ∠ACB = BCD, and CA = CD; therefore if the angle DCB be laid upon BCA, DC will fall upon CA, and D upon A, and conſequently the arch DB will coincide with AB, as well as the ſector DBC with ABC, conſequently arch DB = AB, and ſector DBC = ABC (Ax. 8).

Cor. 1. *In equal circles, the radii making equal angles, comprehend equal arches, and ſectors.*

Cor. 2. *In the ſame or equal circles, the radii making equal angles, comprehend equal cords* AB, BD.

For theſe will coincide with one another. It alſo follows from Prop. VI. II.

Cor. 3. *Equal cords cut off equal arches, and equal ſegments, in the ſame circle.*

For if laid upon one another, they perfectly coincide, as has been proved.

PROP. VII.

In the ſame or equal circles, the arches, and alſo the ſectors, are proportional to the angles intercepted by the radii. 97.

Take any arch AB as ſmall as you will, and let AB = BC, &c. alſo AB = QR = RS, &c. and drawing CA, CB, CD, &c. and PQ, PR, PS, &c. then all the angles ACB, BCD, QPR, RPS, &c. are equal (Cor. 1. Prop. VI). Whence AF is as multiple of AB, as the angle ACF is of ACB. Therefore AB : AF :: ACB : ACF (Prop. V. Proportion). Alſo QV is as multiple of QR or AB, as QPV is of QPR or ACB, whence AB : QV :: ACB : QPV (ibid.); whence AF : QV :: ACF : QPV (Cor. 2. 14. Proportion).

The ſame reaſoning holds in the ſectors, for ſect. ACF is as multiple of ACB; as ∠ACF is of the ∠ACB. And ſect. QPV is as multiple of QPR or

FIG. ABC; as ∠ QPV is of ACB. Therefore ſect.
97. ACF : ſect. QPV : : angle ACF : ∠ QPV.

Cor. *The angle* ACF *is to* 4 *right angles; as the arch* AF, *is to the whole circumference.*

PROP. VIII.

98. *In all circles, ſimilar arches are as the radii of the circles.*

Let the circles AFG and *afg* be both deſcribed from the ſame center, C. Draw the radii CA, CF; then the arches AF, *af* are ſimilar (Def. 18). Draw CB extremely near CA. Then the figures or ſectors C*ab*, CAB, approach very near to iſoſceles triangles, which are ſimilar to one another, becauſe the ∠ at C is common (3. II). Therefore C*a* : *ab* : : CA : AB (13. II); and C*a* : CA : : *ab* : AB (4. Proportion). Now if you ſuppoſe BF divided into more arches, equal to AB; and more radii CB drawn; *bf* will then contain as many arches equal to *ab*. Therefore *af* is as multiple of *ab*, as AF is of AB; therefore *ab* : AB : : *af* : AF (5. Proportion); whence C*a* : CA : : *af* : AF (1. Proportion).

PROP. IX.

98. *The circumferences of circles are to one another, as their diameters.*

For AF : circumference AFGA : ∠ ACF : 4 right angles (Cor. 7) : : ∠ *a*C*f* : 4 right angles : : *af* : circumference *afga*. And AFGA : *afga* : : AF : *af* (4. Proportion) : : CA : *ca* (Prop. VIII) : : 2CA : 2 C*a* (5. Proportion).

Cor. *The circumferences of circles are as their radii.*

PROP. X.

FIG. 88.

A right line AG, *perpendicular to the diameter* AD *of a circle, at the extreme point* A, *touches the circle in that point; and lies wholly without the circle.*

To any point O in the line GAF, draw the line CO from the center. Then the hypothenuſe OC is greater than the ſide AC (4. II). Therefore O is without the circle. And ſo it is for any point beſides A; therefore the line GF is entirely out of the circle.

Cor. 1. *Hence a right line touches a circle only in one point.*

Cor. 2. *If a right line touches a circle in one point, it is perpendicular to the diameter in that point.*

Cor. 3. *All circles, whoſe centers are in the line* AD, *and whoſe circumferences paſs through the point* A, *touch one another, and the line* GAF, *in the ſame point* A.

Cor. 4. *Hence, if two circles touch one another, either inwardly or outwardly; the line paſſing through their centers,* C, B, D, *ſhall alſo paſs through the point of contact,* A. 99.

Otherwiſe a line, touching both circles in that point, could not be perpendicular to both diameters.

Cor. 5. *Two circles, can only touch in one point.*

From the centers B, D, draw BO, DO, to a point O in the exterior circle. Then in the triangle BOD; DB + BO is greater than DO or DA or DB + BA (5. II). Whence BO is greater than BA; therefore the point O, is without the circle AE. In like manner, drawing CO; DO + CO is greater than DA + CA, and CO greater than CA, therefore O falls without the circle AI.

FIG.

PROP. XI.

100. *The angle of contact between a right line and a circle* DAI, *is leſs than any right-lined angle whatever*, DAL.

Draw BE perpendicular to AL, then the ſide BA oppoſite to the right angle BEA, is greater than the ſide BE oppoſite to the acute angle BAE (4. II). Therefore the point E, and ſo the whole line AEL, falls within the circle.

Cor. 1. *Hence the angle of a ſemicircle* BAI *is greater than any acute angle whatever.*

Cor. 2. *The angle of contact* DAI, *is infinitely leſs than a right angle.*

For if it was in a finite proportion to a right angle, then an acute angle might be found equal to it.

Cor. 3. *If any other circle be deſcribed through* A, *with any radius greater than* AB, *it will fall entirely between the tangent* AD *and the circle* AL, *and make the angle of contact leſs. And circles may be deſcribed* ad infinitum, *which ſhall only touch one another in* A; *their centers being all in the line* AB *produced.*

All this appears by Cor. 5. Prop. X. compared with this prop.

PROP. XII.

101.
102. *In a circle, the angle at the center is double the angle at the circumference, ſtanding upon the ſame arch*; $BDC = 2BAC$.

Caſe 1. When one ſide AF paſſes through the center; in the iſoſceles triangle ADC, $\angle DAC = DCA$ (3. II), and the $\angle FDC = DAC + DCA$ (11. II) $= 2FAC$.

Caſe 2. If the center of the circle be within the angle BAC; draw ADF, then by Caſe 1, $FDC = 2FAC$, 2FAC, and FDB = 2FAB, therefore the whole BDC = 2BAC. FIG. 101.

Caſe 3. If the center of the circle be without the angle, BAC; draw ADF, then by Caſe 1, FDB = 2FAB, and FDC = 2FAC, therefore the remainder BDC = 2BAC (Ax. 4). 102.

Cor. 1. *The angle at the circumference ſtanding upon any arch, is equal to half the angle at the center, upon the ſame arch; or to the angle at the center upon half the arch.*

Cor. 2. *In the ſame or equal circles, the angles at the circumference, are equal, which ſtand upon equal arches or equal cords.*

This is plain from Cor. 1, 2. Prop. VI.

PROP. XIII.

All angles in the ſame ſegment of a circle, are equal, DAC = DBC, *and* DGC = DHC. 103.

For ∠DGC and DHC are each equal to the angle at the center, on half the arch DABC. And DAC, DBC are each of them equal to the angle at the center, on half the arch AGHC.

Or thus.

The ∠DGC = ½DOC = DHC (Prop. XII). Again, ∠DEC = DAF + ADF (1. II) = DBC + BCF (ibid.), but ADF = BCF (Prop. XII); therefore DAF or DAC = DBC (Ax. 4).

Cor. *If the extremities of two equal arches* DA, BC, *be joined by right lines,* DC, AB; *they will be parallel.*

For ∠BAC = DCA (Cor. 2. 12), therefore AB, CD are parallel (Cor. 3. 4. I).

PROP.

PROP. XIV.

FIG. 104. *The angle* ABC *in a ſemicircle is a right angle.*

For draw BD to the center, then BDA, BDC are two iſoſceles triangles, therefore $DAB = DBA$, and $DCB = DBC$ (3. II). And $DAB + DCB = DBA + DBC = ABC$ (Ax. 3) = half of two right angles (2. II) = a right angle.

Cor. 1. *The angle* ABG, *in a greater ſegment* ABFG, *is leſs than a right angle; and the angle* ABF, *in a leſs ſegment* ABF, *is greater than a right angle.*

This is evident by inſpecting the figure.

Cor. 2. *If a line be drawn from the middle of the hypothenuſe (of a right-angled triangle), to the right angle; it cuts the triangle into two iſoſceles triangles.*

PROP. XV.

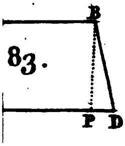
105. *If two lines cutting a circle, interſect one another in* A; *and there be made at the center,* $\angle ECF = BAD$;

Then arch $BD + GH = 2EF$, *if* A *is within the circle; or arch* $BD - GH = 2EF$, *if* A *is without.*

For draw HI parallel to GD, then $DI = GH$ (Cor. 13); and angle $BHI = BAD = ECF$ (4. I). Therefore $EF = \frac{1}{2}BI$ (Cor. 1. 12); and $2EF = BI = BD + GH$, when A is within, but $= BD - GH$, when A is without the circle.

Cor. 1. *If from a point without, two lines touch a circle; the angle made by them is equal to the angle at the center, ſtanding on half the difference, of theſe two parts of the circumference.*

83.



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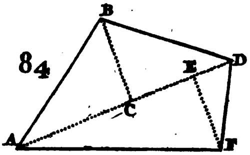

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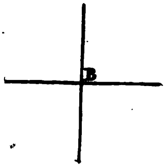

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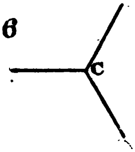

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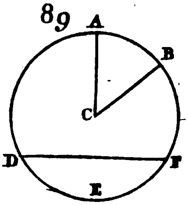

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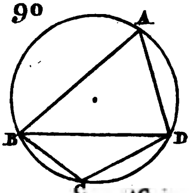

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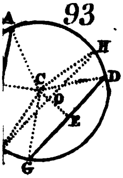

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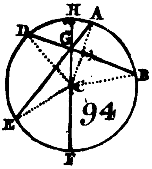

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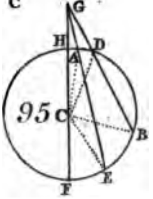

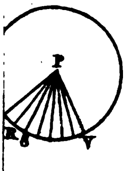

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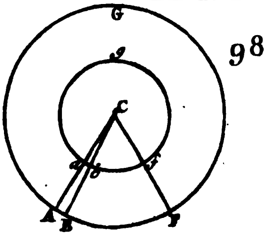

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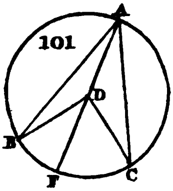

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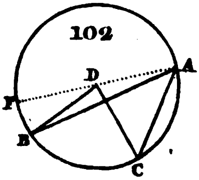

Pl. IV. *Pa. 60.*

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This is plain, by supposing B, H, and G, D to coincide in the periphery, then half their difference will be $= EF$. FIG. 105.

Cor. 2. *The angle* $A = \angle BHD + HDG$, *when* A *is within*; *or* $A = BHD - HDG$, *when* A *is without the circle* (1. II).

PROP. XVI.

In a circle, the angle made at the point of contact between the tangent and any cord, is equal to the angle in the alternate segment; $ECF = EBC$, *and* $ECA = EGC$. 106.

Through the center O, draw the diameter COD, which is $\perp$ to CF (Cor. 2. 10). The $\angle CED$ is right (Prop. XIV); therefore $\angle D + DCE =$ a right angle (Cor. 2. 2. II) $= DCE + ECF$; therefore $D = ECF$, or $EBC = ECF$ (Prop. XIII).

Again, $CEG + ECG + G =$ two right angles (2. II) $= GCF + ECG + ECA$ (1. I), $CEG + G = GCF + ECA$, but $CEG = GCF$ (this prop.), therefore $G = ECA$ (Ax. 4).

Cor. *A tangent to the middle point of an arch, is parallel to the cord of it.*

For if arch $CB = CE$, then cord $CB =$ cord CE (Prop. VI. and Cor. 2); whence $\angle E = B = ECF$ (this prop.), whence BE, CF are parallel (Cor. 3. 4. I).

PROP. XVII.

If from any point B *in a semicircle, a perpendicular* BD *be let fall upon the diameter, it will be a mean proportional between the segments of the diameter:* $AD : DB :: DB : DC$. 107.

For drawing AB, BC, the triangles ABC, ABD, DBC are similar, for $\angle ABC$ is right (Prop. XIV), and

FIG. 111. Cor. 1. *Let* O *be the center, then the rectangle* CFD *is equal to the ſquare of the diſtance from the center — radius ſquare*; $CFD = FO^2 - AO^2$.

For $AF \times FB = \overline{FO - AO} \times \overline{FO + AO} = FO^2 - AO^2$ (12. I).

Cor. 2. *Let* HF *be a tangent at* H; *then the rectangle* CFD = *ſquare of the tangent* FH.

For $FO^2 - AO^2 = FO^2 - OH^2 = FH^2$ (Cor. 1. 21. II).

Cor. 3. *If ſeveral lines* FD, FG, *are drawn from the ſame point* F; *the rectangles of the whole and external ſegment, will be all equal to one another*; CFD = EFG.

For they are all equal to the rectangle AFB.

Cor. 4. *If from the ſame point* F, *two tangents be drawn to the circle, they will be equal*; FH = FI.

For the ſquare of either of them is equal to the rectangle AFB.

PROP. XXII.

112. *If a line* PFC *be drawn perpendicular to the diameter* AD *of a circle; and any line drawn from* A *to cut the circle and perpendicular; then the rectangle of the diſtances of the ſections from* A, *will be equal to the rectangle of the diameter and the diſtance of the perpendicular from* A; $AB \times AC = AP \times AD$.

For draw BD, and the triangles ABD, APC are ſimilar, for ∠ at A is common, and ∠P and B are right (14); therefore AD : AB :: AC : AP (13. II), and $AD \times AP = AB \times AC$ (12. Proportion).

Cor. 1. *If* PF *cuts the circle in* K, *then* $AB \times AC = AK^2$.

Cor. 2. *If more lines* AEF *be drawn, all the rectangles* EAF, BAC *are equal.* FIG. 112.

For they are all equal to the rectangle PAD.

PROP. XXIII.

In a circle EDF *whoſe center is* C, *and radius* CE, *if the points* B, A, *be ſo placed in the diameter produced, that* CB, CE, CA *be in continual proportion, then two lines* BD, AD *drawn from theſe points, to any point in the circumference of the circle, will always be in the given ratio of* BE *to* AE. 113.

Draw DP perpendicular to the diameter EF, then $DP^2 = EP \times PF$ (17) $= 2CE \times EP - EP^2$, whence $AD^2 = \overline{AE + EP}^2 + PD^2$ (21. II) $= AE^2 + EP^2 + 2AEP + 2CEP - EP^2$ (10. I) $= AE^2 + 2CE \times EP + 2AE \times EP$. Alſo $BD^2 = \overline{BE - EP}^2 + PD^2$ (21. II) $= BE^2 - 2BEP + EP^2 + 2CEP - EP^2$ (11. I) $= BE^2 + 2CE \times EP - 2BE \times EP$.

And ſince CA : CE : CB $\div\div$, therefore AE : CE :: EB : CB (13. Proportion), or AE : EB :: CE : CB (4. Proportion). Alſo $AE^2 : EB^2 :: CE^2 : CB^2 :: CA : CB$ (23. Proportion) $:: CE + AE : CE - EB :: 2CE \times EP + 2AE \times EP : 2CE \times EP - 2EB \times EP$ (5. Proportion). And $AE^2 : EB^2 :: AE^2 + 2CE \times EP + 2AE \times EP : EB^2 + 2CE \times EP - 2EB \times EP$ (10. Proportion) $:: AD^2 : BD^2$. And AE : EB :: AD : BD (Cor. 3. 18. Proportion).

PROP. XXIV.

If D, C *be two points in the diameter of a circle, equidiſtant from the center* O; *and if two lines be drawn from thence to any point* E, *in the circumference, the ſum of their ſquares will be equal to the ſum of the ſquares of the ſegments of the diameter*; $DE^2 + CE^2 = AC^2 + CB^2$. 114.

FIG. 114. For draw EO to the center O, then $DE^2 + CE^2 = 2DO^2 + 2OE^2$ (28. II) $= 2AO^2 + 2OC^2$. But $AC^2 + CB^2 = \overline{AO + OC}^2 + \overline{AO - OC}^2 = AO^2 + OC^2 + 2AOC + AO^2 + OC^2 - 2AOC$ (10. II) $= 2AO^2 + 2OC^2 = DE^2 + CE^2$.

Cor. 1. *Hence the ſum of the ſquares of* DE, CE *is equal to twice the ſquare of the radius* + *twice the ſquare of the diſtance of one of the points from the center*; $DE^2 + CE^2 = 2AO^2 + 2OC^2$.

Cor. 2. *The ſum of the ſquares of any two correſpondent ones will be equal.*

For they are all equal to the ſame given quantity.

PROP. XXV.

115. *If any cord* PQ *be drawn parallel to the diameter* AB, *of a circle; and from a given point* C *in that diameter, the lines* CP, CQ *be drawn to the two ends of the cord; I ſay the ſum of their ſquares is equal to the ſum of the ſquares of the ſegments of the diameter*; $CP^2 + CQ^2 = AC^2 + CB^2$.

For draw PS, QR ⊥ to the diameter AB, then PS^2 or $QR^2 = PC^2 - SC^2 = QC^2 - RC^2$ (21. II); that is, $PC^2 - \overline{SO + OC}^2 = QC^2 - \overline{SO - OC}^2$; or $PC^2 - SO^2 - 2SOC - OC^2$ (10. I) $= QC^2 - SO^2 + 2SOC - OC^2$, becauſe $OQ = OS$. Therefore $PC^2 = QC^2 + 4SOC$, but $AC^2 + CB^2 = \overline{AO + OC}^2 + \overline{AO - OC}^2 = 2AO^2 \times 2OC^2$ (10, 11. I). But $PC^2 = AO^2 + OC^2 + 2SOC$ (22. II) $= QC^2 + 4SOC$. Therefore

$$QC^2 = AO^2 + OC^2 - 2SOC.$$
$$PC^2 = AO^2 + OC^2 + 2SOC$$

therefore $PC2 + QC^2 = 2AO^2 + 2OC^2 = AC^2 + CB^2$.

Cor. 1. *The ſum of their ſquares*, $PC^2 + QC^2 = 2AO^2 + 2OC^2$. FIG. 115.

Cor. 2. *The difference of their ſquares*, $PC^2 - QC^2 = 4SOC$.

Cor. 3. *All theſe things hold good, if the point* C *is taken without the circle.*

PROP. XXVI.

In a circle, if a perp. DB *be let fall from any point* D, *upon the diameter* CI, *and the tangent* DO *drawn from* D; *then* AB, AC, AO, *will be continually proportional.* 116.

Draw the radius DA, then the triangles ABD, ADO, are ſimilar, for the angles at B and D are right (Cor. 2. 10), and angle A common; whence AB : AD : : AD : AO; that is, AB : AC : AO ∺.

PROP. XXVII.

If a triangle ADC *be inſcribed in a circle; and if* BC *be drawn parallel to the tangent* AT; *then* AB, AC, AD, *are continually proportional.* 117.

For the triangle ABC, is ſimilar to ACD; for ∠ D = TAC (16) = ACB (4. II), and A is common; therefore AB : AC : : AC : AD (13. II).

Cor. AD : DC : : AC : CB.

PROP. XXVIII.

If a triangle BDF *be inſcribed in a circle, and a perpendicular* DP *let fall from* D *on the oppoſite ſide* BF, *and the diameter* DA *drawn; then as the perpendicular, is to one ſide including the angle* D; *ſo the other ſide, to the diameter of the circle*; DP : DB : : DF : DA. 118.

FIG. 118. For drawing AF, the triangles BDP, and ADF are ſimilar; for $\angle A = B$ (13), and angles at P and F are right (14); therefore DP : DB : : DF : DA (13. II).

Cor. *The rectangles of the ſides of an inſcribed triangle; is equal to the rectangle of the diameter, and the perp. on the third ſide.*

PROP. XXIX.

119. *If a triangle* BAC *be inſcribed in a circle, and the angle* A *biſected by the right line* AED; *then as one ſide, to the ſegment of the biſecting line, within the triangle; ſo the whole biſecting line, to the other ſide*; AB : AE : : AD : AC.

Draw BD, then the triangles ABD, ACE are ſimilar; for $\angle D = C$ (Cor. 2. 12), and BAD = EAC (hyp.); therefore AB : AD : : AE : AC (13. II); and AB : AE : : AD : AC (4. Proportion).

Cor. *If an angle of a triangle (inſcribed in a circle) be biſected; the rectangle of the ſides, is equal to the rectangle of the whole biſecting line within the circle, and the ſegment within the triangle*: BAC = DAE.

PROP. XXX.

120. *If a circle be inſcribed in a triangle* ABC, *and lines be drawn from the center* D, *to the points of contact* E, F, G; *then any ſegment* BF *or* BE *joining to the angle* B, *is equal to half the ſum of the three ſides — the oppoſite ſide* AC.

For the triangles BDF, BDE are ſimilar and equal (9. II); for $\angle F = \angle E$ a right one (10), and DE = DF, and BD common; whence BF = BE. In like manner CF = CG, and AE = AG. Then ſince the ſum of the ſides is BC + CA + AB =

2BF + 2CG + 2AG, therefore half the ſum = FIG.
BF + CG + AG = BF + AC, therefore BF = 120.
½ ſum — AC.

Cor. *The area of the triangle* BAC, *is equal to the rectangle of the radius* DF, *and half the ſum of the three ſides.*

For the triangle ABC is made up of the three triangles ADB, BDC, CDA, whoſe common hight is the radius DF.

PROP. XXXI.

If a quadrilateral ABCD *be inſcribed in a circle, the* 121.
ſum of two oppoſite angles is equal to two right angles;
ADC + ABC = *two right angles.*

Draw AC, BD, and produce AB to E; then the external angle CBE = BCA + BAC (1. II) = BDA + BDC (13) = ADC; therefore CBE + CBA = ADC + CBA = 2 right angles (1. I).

Cor. *If one ſide of a quadrangle (inſcribed in a circle) be produced, the external angle* EBC *is equal to the internal oppoſite angle* ADC.

PROP. XXXII.

If a quadrangle be inſcribed in a circle; the rectangle 122.
of the diagonals, is equal to the ſum of the rectangles of
the oppoſite ſides; AC × BD = AB × CD + AD
× BC.

Make the angle ABF = CBD, then ABD = CBF; and ſince the ∠CDB = FAB (13), the triangles FAB, and CDB are ſimilar, whence DC : DB :: AF : AB (13. II), and CD × AB = BD × AF (12. Proportion). Alſo ſince ∠BCF = BDA (13), the triangles CBF and DBA are ſimilar; whence CB : CF :: DB : DA (13. II), and CB

FIG. 122. $\times DA = BD \times CF$ (12. Proportion). Therefore $CD \times AB + CB \times DA = BD \times AF + BD \times CF = BD \times AC$ (Ax. 3).

PROP. XXXIII.

A circle is equal to a triangle whoſe baſe is the circumference of the circle; and hight, its radius.

123. Let AB be equal to the length of the circumference, and let the circle touch it in I; draw CI, and CD extremely near it. Then by reaſon of the extreme ſmallneſs of the arch DI, the ſector CD coincides with the triangle CDI, and the arch with a portion of the right line. Now ſince the circle DEGF may be ſuppoſed to be made up of ſuch ſectors CDI, and the triangle ACB of as many triangles CDI equal to the ſector CDI; it follows that all theſe ſectors are equal to all the triangles, or the circle DEGF = the triangle ABC.

This is alſo evident by the 25. III, for a circle may be conſidered as a regular polygon of an infinite number of ſides, whoſe hight is the radius of the circle.

Cor. *The ſector of a circle is equal to a triangle, whoſe baſe is the arch, and hight the radius.*

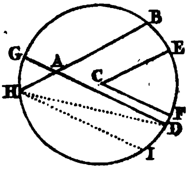
PROP. XXXIV.

123. *The area of a circle is equal to the rectangle of half the circumference and half the diameter.*

For the circle EGF is equal to the triangle ABC (33), and that triangle is equal to the rectangle of half the baſe AB and hight CI, that is, of half the circumference DFGED, and half the diameter CI (Cor. 2. 10. II).

Cor. 1. *The ſector of a circle is equal to the rectangle of half the arch and the radius.*

Cor.



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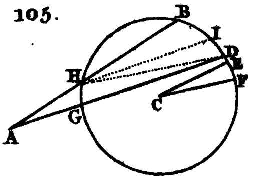

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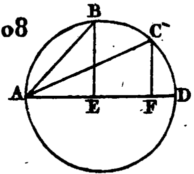

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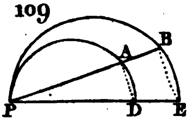

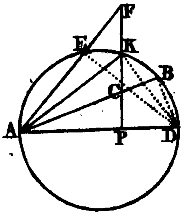

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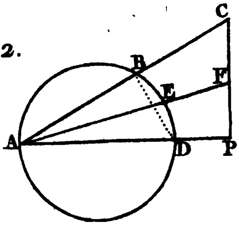

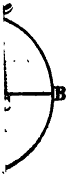

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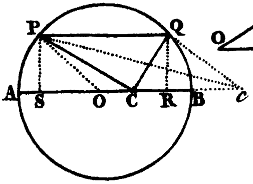

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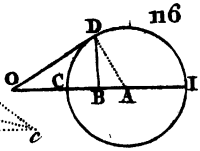

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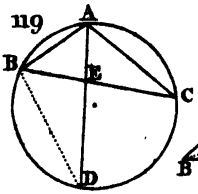

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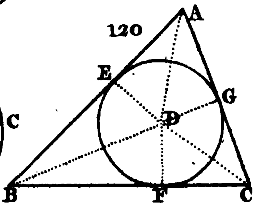

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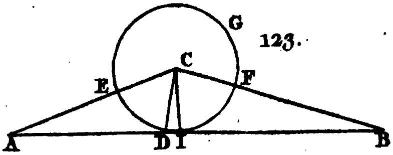


Pl. V. *Pa. 70.*

Cor. 2. *Sectors are to one another in the complicate ratio of their arches and radii.* FIG. 123.

For triangles, to which they are equal, are in that ratio (Cor. 1. 11. II).

PROP. XXXV.

Circles (that is, their areas) are to one another as the ſquares of their diameters.

For circumference EFGE : cir. IHKI : : AB : CD (9); and the areas of the circles EFG, and IHK are $\frac{EFG \times AB}{4}$, and $\frac{IHK \times CD}{4}$ (34); therefore circle EF : circle IH : : EFGE $\times$ AB : IHKI $\times$ CD (5. Proportion) : : AB^2 : CD^2 (7. Proportion). 124.

Cor. 1. *Circles are to one another as the ſquares of the radii, or as the ſquares of the circumferences.*

For theſe are all in the ſame ratio (Prop. IX).

Cor. 2. *A circle made on the hypothenuſe, is equal to two circles made alike on the two ſides, of a right-angled triangle.*

PROP. XXXVI.

Similar polygons deſcribed in circles, are to one another, as the circles wherein they are inſcribed.

Draw CK, AG, then becauſe ſimilar polygons may be reſolved into ſimilar triangles (Cor. 2. 19. III), therefore AF : AG : : CH : CK, and AG : AB : : CK : CD (13. II), therefore AF : AB : : CH : CD. Or at once, AF : AB : : CH : CD (19. III). And polygon EF : polygon IH : : AF^2 : CH^2 : : AB^2 : CD^2 (20. III) : : circle EF : circle IH (35). 124.

FIG. 127. For $\frac{AB \times AD}{2R}$ = perp. from A upon BD (28), and $\frac{BC \times CD}{2R}$ = perp. from C on BD (ibid.), and $\frac{AB \times AD + BC \times CD}{2R} = \frac{T}{2R}$ is the ſum of the perpendiculars.

Therefore $\frac{T \times BD}{2R}$ = twice the area of the trapezium (24. III); and in like manner $\frac{AB \times BC + AD \times DC}{2R} \times AC$ or $\frac{P \times AC}{2R}$ = twice the ſame area. Therefore $T \times BD = P \times AC$, and $AC = \frac{T \times BD}{P}$. But $AC \times BD = AB \times CD + AD \times CB$ (32) $= Q = \frac{T \times BD^2}{P}$, and $BD^2 = \frac{PQ}{T}$, and $BD = \sqrt{\frac{PQ}{T}}$. Whence $\frac{T \times BD}{2R} = \frac{T}{2R}\sqrt{\frac{PQ}{T}} = \frac{\sqrt{PQT}}{2R}$ = twice the area; and the area $= \frac{\sqrt{PQT}}{4R}$.

Cor. 1. $BD^2 = \frac{PQ}{T}$, and $AC^2 = \frac{QT}{P}$.

Cor. 2. $BD : AC :: P : T :: AB \times BC + CD \times DA : AB \times AD + BC \times CD$.

PROP. XL.

127. *If* ABCD *be a trapezium inſcribed in a circle, and each ſide be ſubtracted from the ſum of the other three, there will be four remainders; then take the product of two of theſe remainders, and likewiſe the product of the other two; I ſay, 4 times the area of the trapezium will be a mean proportional, between theſe two products.*

From A let fall the perpendiculars AF, AP upon CB, CD. Then ſince ∠ABF = ADC (Cor. 31); the firſt perp. falls without, and the ſecond within the figure. Alſo the angles B, P, being right, the triangles ABF, APD, are ſimilar. Therefore AB : BF : : AD : DP $= \frac{AD \times BF}{AB}$ (13. II), and AF $= \sqrt{AB^2 - FB^2}$ (21. II), alſo AB : AF : : AD : AP $= \frac{AD}{AB}\sqrt{AB^2 - FB^2}$. Draw AC, then $AB^2 + BC^2 + 2BC \times BF = AC^2 = AD^2 + CD^2 - 2CD \times \frac{AD \times BF}{AB}$ (22, 23. II). Whence $2BC \times BF + \frac{2CD \times AD}{AB} \times BF = AD^2 + CD^2 - AB^2 - BC^2$, and $\frac{BF}{AB} = \frac{AD^2 + CD^2 - AB^2 - BC^2}{2AB \times BC + 2CD \times DA}$ $= \frac{AD^2 + CD^2 - AB^2 - BC^2}{2P}$ (putting P = AB × BC + CD × DA). And $\frac{AB + BF}{AB} =$ $\frac{CD^2 + 2CD \times AD + AD^2 - AB^2 + 2AB \times BC - BC^2}{2P}$ $= \frac{\overline{CD + AD}^2 - \overline{AB - BC}^2}{2P}$, and $\frac{AB - BF}{AB}$ $= \frac{AB^2 + 2AB \times BC + BC^2 - CD^2 + 2CD \times AD - AD^2}{2P}$ $= \frac{\overline{AB + BC}^2 - \overline{CD - AD}^2}{2P}$. FIG. 127.

But twice the area $= AF \times BC + AP \times CD$ (24. III) $= BC\sqrt{AB^2 - BF^2} + \frac{CD \times AD}{AB}\sqrt{AB^2 - BF^2}$ $= \frac{AB \times BC + CD \times AD}{AB}\sqrt{AB^2 - BF^2} =$ $\frac{P}{AB}\sqrt{AB^2 - BF^2}$; and 4 × area ſquare = $PP \times \frac{\overline{AB^2 - BF^2}}{AB^2} = PP \times \frac{AB + BF}{AB} \times \frac{AB - BF}{AB}$ =

FIG. 127. $= PP \times \frac{\overline{CD+AD}^2 - \overline{AB-BC}^2}{2P} \times \frac{\overline{AB+BC}^2 - \overline{CD-AD}^2}{2P}$, and $16 \times \overline{\text{area}}^2 = \overline{\overline{CD+AD}^2 - \overline{AB-BC}^2} \times \overline{\overline{AB+BC}^2 - \overline{CD-AD}^2}$ $= \overline{CD+AD+AB-BC} \times \overline{CD+AD-AB+BC} \times \overline{AB+BC+CD-AD} \times \overline{AB+BC+AD-CD}$ (12. I).

Cor. *If* S = *half the ſum of the four ſides, then the area is a mean proportional between theſe rectangles* $\overline{S-AB} \times \overline{S-BC}$, *and* $\overline{S-CD} \times \overline{S-DA}$.

For $\text{area}^2 = \frac{CD+AD+AB-BC}{2} \times$ &c. but $S - AB = \frac{CD+AD+AB+BC}{2} - AB = \frac{CD+AD-AB+BC}{2}$, and ſo of the reſt.

PROP. XLI.

128. *If an equilateral triangle* ABC *be inſcribed in a circle; the ſquare of the ſide thereof, is equal to three times the ſquare of the radius:* $AB^2 = 3AD^2$.

Draw the diameter AE, and the cord BE. Then the triangle BDE is equiangular (Cor. 1. 2. II), for ∠BDE = BAC (Cor. 1. 12) = BCA = BED, and BE = DB (Cor. 1. 3. II). Then $AB^2 + BE^2 = AE^2$ (21. II) $= 4DB^2 = 4BE^2$, and $AB^2 = 3BE^2 = 3BD^2$.

Cor. 1. $AB^2 : AF^2 :: 4 : 3$.

For $AB^2 : AF^2 :: AE : AF$ (23. Proportion) $:: 4 : 3$.

Cor. 2. DF = *half* DE.

Cor. 3. *The ſide* BC *of the equilateral triangle, cuts off a fourth part of the diameter.*

PROP.

PROP. XLII.

FIG. 129.

A ſquare inſcribed in a circle, is equal to twice the ſquare of the radius; $AB^2 = 2BO^2$.

For $AB^2 = AO^2 + OB^2$ (21. II) $= 2AO^2$.

Cor. *The circumſcribed ſquare* EG *is double the inſcribed ſquare*, AC.

For EG is the ſquare of the diameter or 4 ſquares of the radius, and therefore equal to two of the inſcribed ſquares, ABCD.

PROP. XLIII.

130.

If two diagonals BD, EC *be drawn to cut one another, in an inſcribed regular pentagon. The greater ſegments* EF, BF, *will be equal to the ſide of the pentagon*, AB.

For ſince the arch AE = BC, and AB = ED, therefore EC is parallel to AB, and BD parallel to AE (Cor. 13); therefore ABFE is a parallelogram, and EF = AB = AE = BF (1. III).

Cor. 1. *The diagonals* BD, CE *cut one another in extreme and mean proportion*; BD : BF : : BF : FD.

For ∠DCF = CDF = CBD (Cor. 2. 12); therefore the triangles CDF, CDB are ſimilar, BD : DC : : DC : DF (13. II); that is, BD : BF : : BF : FD.

Cor. 2. *The diagonal* CE *is parallel to* AB, *and* BD *to* AE.

Cor. 3. *The ſide of the pentagon* BC, *is to the diagonal* BD, *as* 1 *to* $\frac{1+\surd 5}{2}$.

For

FIG. 130. For $BD \times FD$ or $BD \times \overline{BD - BC} = BF^2$ (Cor. 1); that is, $BD^2 - BD \times BC = BC^2$; add $\frac{1}{4}BC^2$, then $BD^2 - BD \times BC + \frac{1}{4}BC^2 = \frac{5}{4}BC^2$; that is, $\overline{BD - \frac{1}{2}BC}|^2 = 5 \times \frac{BC^2}{4}$, and the root is $BD - \frac{1}{2}BC = \frac{BC}{2}\sqrt{5}$, and $BD = \frac{BC}{2} + \frac{BC}{2}\sqrt{5} = BC \times \frac{1 + \sqrt{5}}{2}$.

Cor. 4. *The angle* BCF *is double to the angle* CBF.

For it ſtands on double the arch.

PROP. XLIV.

130. *If a regular pentagon be inſcribed in a circle; the ſquare of the radius* AH, *is to the ſquare of its ſide,* AB; *as* 2 *to* $5 - \sqrt{5}$.

Let HG biſect AB in I, and make $IO = IG$. Then the angles AIO, AIG are right (Cor. 3. 3. II). And put $R =$ radius AH. The $\angle GHA = \frac{2}{5}$ of a right angle (1. I), and $IAH = \frac{3}{5}$ of a right angle (Cor. 2. 2. II); but IAG or $IAO = \frac{1}{2}GHA$ (12) $= \frac{1}{5}$ of a right angle, therefore $OAH = \frac{2}{5}$ of a right angle (Ax. 4) $= OHA$, whence $HO = OA = AG$. $2R \times GI = GA^2$ (Cor. 17) $= HO^2 = \overline{R - 2GI}|^2 = RR - 4R \times GI + 4GI^2$ (11. I), and $4GI^2 - 6R \times GI + RR = 0$ (Ax. 3. 4), and $GI^2 - \frac{3}{2}R \times GI + \frac{1}{4}RR = 0$ (Ax. 6); add $\frac{5}{16}RR$, then $\frac{9}{16}RR - \frac{3}{2}R \times GI + GI^2 = \frac{5}{16}RR$; that is, $\overline{\frac{3}{4}R - GI}|^2 = 5 \times \frac{RR}{16}$, whence the root $\frac{3}{4}R - GI = \frac{R}{4}\sqrt{5}$, and $GI = \frac{3 - \sqrt{5}}{4}R$. But $\frac{3}{2}R \times GI - GI^2 = \frac{1}{4}RR$. And $AI^2 = 2R \times GI - GI^2$ (17) $= \frac{1}{2}R \times GI + \frac{3}{2}R \times GI - GI^2 = \frac{1}{2}R \times GI + \frac{1}{4}RR = \frac{1}{2}R \times \frac{3 - \sqrt{5}}{4}R + \frac{RR}{4} =$

$RR \times \frac{5-\sqrt{5}}{8}$, and $4AI^2$ or $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$. FIG. 130.

Cor. 1. *The ſquare of the perpendicular* HI, *upon the ſide of the pentagon, is equal to* $\frac{3+\sqrt{5}}{8}RR$.

For $HI^2 = R^2 - AI^2 = \frac{8R^2 - 5R^2 + \sqrt{5}\,RR}{8} = \frac{3+\sqrt{5}}{8}RR$.

Cor. 2. *The ſquare of radius* AH, *is to the ſquare of the diagonal* BD, *as* 1 *to* $\frac{5+\sqrt{5}}{2}$.

For BC or $AB = \frac{2BD}{1+\sqrt{5}}$ (Cor. 3. 43), and $AB^2 = \frac{4BD^2}{\overline{1+\sqrt{5}}^2} = \frac{4BD^2}{1+5+2\sqrt{5}}$ (10. I) $= \frac{4BD^2}{6+2\sqrt{5}}$, and $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$ (44), therefore $\frac{4BD^2}{6+2\sqrt{5}}$ or $\frac{2BD^2}{3+\sqrt{5}} = RR \times \frac{5-\sqrt{5}}{2}$, and $2BD^2 = RR \times \frac{5-\sqrt{5}}{2} \times \overline{3+\sqrt{5}} = RR \times \frac{15+5\sqrt{5}-3\sqrt{5}-5}{2}$ (Cor. 1. 8. I) $= RR \times \frac{10+2\sqrt{5}}{2}$, that is, $BD^2 = RR \times \frac{5+\sqrt{5}}{2}$.

Cor. 3. *If* CE *be the diagonal of the pentagon, and* OLD *be drawn; then* $DL = R \times \frac{5-\sqrt{5}}{4}$. 134.

For $DL = \frac{DE^2}{DF}$ (Cor. 17) $= RR \times \frac{5-\sqrt{5}}{4R} = R \times \frac{5-\sqrt{5}}{4}$ (44).

PROP.

FIG.

PROP. XLV.

131. *The ſide of a regular hexagon inſcribed in a circle, is equal to the radius of the circle;* BE = BD.

For ∠BDE = $\frac{1}{6}$ of four right angles (Cor. 1. I) = $\frac{1}{3}$ of two right angles. And the angles B and E together = $\frac{2}{3}$ of two right angles (2. II), whence BED = $\frac{1}{3}$ of two right angles (3. II) = BDE; therefore BE = BD (Cor. 1. 3. II).

PROP. XLVI.

132. *The ſquare of the ſide of a regular octagon, inſcribed in a circle, is equal to the ſquare of half the ſide of the inſcribed ſquare, together with the ſquare of the difference of that half ſide and the radius;* $AB^2 = AP^2 + \overline{OB - AP}^2$.

For $AB^2 = AP^2 + PB^2$ (21. II), but ∠PAO = POA (Cor. 1. 12); therefore PO = AP, and BP = OB − OP = OB − AP; and $AB^2 = \overline{OB - AP}^2 + AP^2$.

Cor. *The ſquare of radius, is to the ſquare of the ſide of the octagon; as* 1 *to* $2 - \sqrt{2}$.

For $AP = \sqrt{\frac{1}{2}AO^2}$, and $PB^2 = \overline{BO - AP}^2 = \overline{BO - AO\sqrt{\frac{1}{2}}}^2 = BO^2 - \frac{2BO \times AO}{\sqrt{2}} + \frac{AO^2}{2} = 1\frac{1}{2}AO^2 - AO^2\sqrt{2}$, add $AP^2 = \frac{1}{2}AO^2$, and $AP^2 + BP^2$ or $AB^2 = 2AO^2 - AO^2\sqrt{2}$.

PROP. XLVII.

133. *The radius of a circle is a mean proportional, between the ſide of an inſcribed regular decagon, and the ſum of that ſide and the radius;* AB : DA :: DA : DA + AB.

Pro-

FIG. 133.

Produce AB to F, ſo that BF may be $= BD$; and draw DF, DB. Then $\angle ADB = \frac{1}{5}$ of two right angles, and therefore DAB and DBA together $= \frac{4}{5}$ of two right angles (2. II), and ABD $= \frac{2}{5}$ of two right angles (3. II) $= BDF + BFD$ (1. II) $= 2BDF$ (3. II); therefore BDF or BFD $= \frac{1}{5}$ of a right angle $= ADB$. Therefore the triangles ADB, and ADF are ſimilar, for $F = ADB$; and A is common, whence AF or $AB + BD : AD :: AD : AB$.

Cor. 1. *If the radius be cut in extreme and mean ratio, the greater ſegment is the ſide of the decagon,* AB.

For ſince $AB + AD : AD :: AD : AB$. Therefore $AB : AD :: AD - AB : AB$ (13. Proportion), or $AD : AB :: AB : AD - AB$, therefore AD is cut in extreme and mean proportion (Def. 11. Proportion).

Cor. 2. *The radius is to the ſide of the decagon; as* 2 *to* $\sqrt{5} - 1$.

For $AB^2 + AB \times AD = AD^2$ (12. Proportion), add $\frac{1}{4}AD^2$, then $AB^2 + AB \times AD + \frac{1}{4}AD^2 = \frac{5}{4}AD^2$ (Ax. 3); and $AB + \frac{1}{2}AD = \frac{AD}{2}\sqrt{5}$; and $AB = \frac{AD}{2}\sqrt{5} - \frac{AD}{2}$, and $2AB = AD \times \overline{\sqrt{5} - 1}$.

Cor. 3. *The ſquare of the perpendicular upon the ſide of a decagon, is* $\frac{5 + \sqrt{5}}{8}$ × *the ſquare of the radius.*

For $\frac{1}{2}AB = \text{rad.} \times \frac{\sqrt{5} - 1}{4}$, and its ſquare =

FIG. 133. $AD^2 \times \frac{3-\sqrt{5}}{8}$, and the ſquare of the perpend. $= AD^2 - AD^2 \times \frac{3-\sqrt{5}}{8} = AD^2 \times \frac{5+\sqrt{5}}{8}$.

PROP. XLVIII.

134. *The ſquare of the ſide of a regular pentagon inſcribed in a circle, is equal to the ſum of the ſquares of the radius, and of the ſide of a regular decagon, inſcribed in the ſame circle;* $AB^2 = FA^2 + AO^2$.

Draw OG perpendicular to the cord FA, to cut it in G, and draw FH. The triangles ABO, HBO are ſimilar; for ∠AOB = $\frac{1}{5}$ of 4 right angles, or $\frac{2}{5}$ of 2 right angles (1. I), alſo BAO and ABO are together = $\frac{3}{5}$ of 2 right angles (2. II), and therefore BAO = $\frac{3}{10}$ of 2 right angles, but BOG (= $\frac{3}{4}$ AOB) = $\frac{3}{4} \times \frac{2}{5}$ of 2 right angles = $\frac{3}{10}$ of 2 right angles, therefore BAO = BOG, and B is common; whence $AB : BO :: BO : BH = \frac{BO^2}{AB}$.

Again, the triangles AFH and ABF are ſimilar, for ∠A = AFH (3. II), and ∠A = B (Cor. 2. 12), therefore $BA : AF :: AF : AH = \frac{AF^2}{AB}$; therefore $AB = AH + BH = \frac{AF^2 + BO^2}{AB}$, or $AB^2 = BO^2 + AF^2$.

Cor. *The perpendicular* OI *upon the ſide of the pentagon, is equal to half the ſum of the radius and ſide of the decagon;* $OI = \frac{AO + AF}{2}$.

For $OI = OF - FI = \frac{2OF^2 - 2OF \times FI}{2OF} = \frac{2OF^2 - FA^2}{2OF}$ (Cor. 17). And ſince $AO^2 = FA^2 +$

AO × AF (47), therefore $AO^2 - FA^2 = AO \times AF$, and $2FO^2 - FA^2 = FO^2 + AF \times FO$, and $\frac{AF + FO}{2} = \frac{2OF - FA^2}{2OF} = OI$. FIG. 134.

PROP. XLIX.

135. *The ſide of a regular dodecagon inſcribed in a circle, is a mean proportional between the radius, and the exceſs of the diameter above the ſide of the inſcribed equilateral triangle.*

Let AB be a ſide of the dodecagon, and draw CB, CF, and DF the ſide of the triangle, and FR perpendicular to AC. Then ACF = $\frac{1}{3}$ of 2 right angles = CAF = CFA (2. II), therefore ACF is an equilateral triangle, and $AO = \frac{1}{2}AC$, and $CO = \sqrt{\frac{3}{4}AC^2}$ (Cor. 39. II), and $BO = CA - CO = CA - \sqrt{\frac{3}{4}CA^2}$, and $BO^2 = CA^2 + \frac{3}{4}CA^2 - 2CA \times \sqrt{\frac{3}{4}CA^2} = 1\frac{3}{4}CA^2 - CA^2\sqrt{3}$ (11. I), and $AB^2 = AO^2 + OB^2$ (21. III) $= \frac{1}{4}AC^2 + 1\frac{3}{4}AC^2 - CA^2\sqrt{3} = 2AC^2 - CA^2\sqrt{3}$. Therefore CA : AB :: AB : $2CA - \overline{CA\sqrt{3}}$. But 2CA is the diameter, and $CA \times \sqrt{3}$ = ſide DF of the equilateral triangle (41).

Cor. *The ſide of the dodecagon*, $AB = CA \times \sqrt{2 - \sqrt{3}}$.

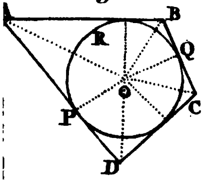
PROP. L.

136. *If* ABH *be an equilateral triangle, and* APDFG *an equilateral pentagon, inſcribed in a circle, both placed with their angles at* A; *then the cord* BD *will be the ſide of an equilateral quindecagon; and* BI *will be half the difference of the ſides of the triangle and pentagon, and* DI (*perp. to* BH) *is the difference of the perpendiculars in the two figures; and* $BD^2 = BI^2 + DI^2$.

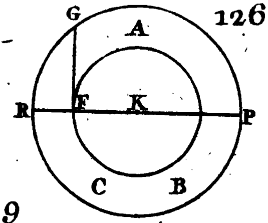
For

FIG. 136. For arch AP $= \frac{1}{5} = \frac{3}{15}$ the circumference, and APD $= \frac{6}{15}$ of it; alſo APB $= \frac{1}{3} = \frac{5}{15}$ the circumference; therefore APD — APB $= \frac{1}{15}$ the circumference; and the cord BD, the ſide of the quindecagon; moreover DI = CO — CE, is known from Cor. Prop. XLIV. and Cor. 2. Prop. XLI. Alſo BI $= \frac{BH - DF}{2}$, which will be known by Prop. XLI, and Prop. XLIV; and thence $BD^2 = BI^2 + DI^2$, will be known.

125



126



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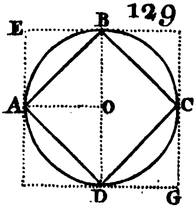

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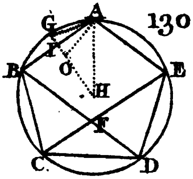

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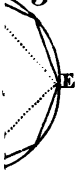

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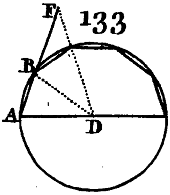

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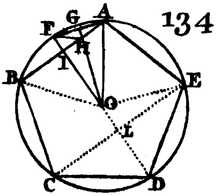
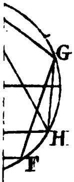




Pl.VI. *Pa.* 84

BOOK V.

Of Planes, and ſolid Angles.

DEFINITIONS.

FIG.

1. A *Plane* is a ſurface which lies even between the extremes; or in which all right lines coincide.

2. A *curve ſurface*, is that whoſe parts lie not even between the extremes; but gradually riſe or fall.

3. A *convex ſurface*, is that which ſwells or riſes up towards the middle.

4. A *concave ſurface*, is that whoſe middle parts are hollow, or fall lower than the extremes.

5. A *right line*, is ſaid to be *perpendicular* to a plane, when it is perpendicular to all lines, drawn to the foot of it, as AB. 140.

6. The *common ſection* of two planes, is the line made by two planes cuting each other, as EF. 139.

7. One *plane* is ſaid to be *perpendicular* to another, when it paſſes through a right line which is perpendicular to the other, as CD. 144.

8. The *inclination* of a *line* to a plane, or the angle it makes with it, is the angle that line makes with a line drawn from the foot of it to the point where a perpendicular, let fall from the top, cuts the plane, as FCI. 142.

9. The *inclination* of two planes, is the angle made by two right lines; both drawn perpendicular to the common ſection, from any point there- 149.

FIG. 149. in; as ∠BFD is the inclination of the planes, AB, CD.

This is the angle which the planes make with one another.

145. 10. *Parallel planes* are thoſe which are every where at the ſame diſtance from each other, as AC, EG.

150. 11. A *ſolid angle* is a ſpace bounded by ſeveral plane angles, meeting in one point, called the *angular point or vertex*, as A.

PROP. I.

137. *If a right line in a plane be produced, it will ſtill be in that plane.*

For produce BC in the plane AD, directly to E; and if BCF be alſo a right line, then BCE and BCF are both right lines; and you have two right lines BCF, BCE, with the part BC the ſame in both; contrary to the nature of a right line (Def. 5. I).

Cor. 2. *If two diſtant points of a right line be in a plane, all the line is ſo.*

PROP. II.

138. *If two right lines* GH, IK, *interſect one another, they are in the ſame plane.*

For imagine A, and B, to be in one plane with C; then the line IACK, as alſo HBCG, and all the points between A and B, are in one plane (Prop. I. and Cor.). And the like for all other lines as AB, drawn between GCH, and ICK.

Cor. *Every part of a triangle is in the ſame plane.*

PRO-

PROP. III.

FIG.

If two planes AB, CD, *cut one another; their* 139.
common ſection EF *is a right line.*

For draw the line EF in the plane AB, then any point G of that right line, is in the plane AB (1); but becauſe the line EF (drawn between the two points E, F, in the plane CD) is alſo in the plane CD; any point G of that line will be in the plane CD. Therefore G being in both planes, will be in their common ſection; and their common ſection EGF is conſequently, a right line.

PROP. IV.

If a right line AB *be perpendicular to two lines* IK, 140.
GH, *at the point of interſection* B; *then is the line* AB *perpendicular to the plane* FD *paſſing through them.*

Let BI = BG = BK = BH, and draw GI, HK, LM, AI, AL, AG, AH, AM, AR. The triangles ABI, ABG, ABH, ABK, are all equal (6. II), and AI = AG = AH = AK. Alſo the triangles GBI, HBK are equal (6. II), and GI = KH, ∠G = ∠H. Alſo the triangles LBG, MBH are equal (7. II), and BL = BM, and GL = HM. And the triangles AGI, AKH are equal (8. II); and conſequently ∠AGI = ∠AHM; whence the triangles AGL, AHM are equal (6. II), and AL = AM. Alſo the triangles ABL, ABM are equal (8. II), and ∠ABL = ABM = a right angle. And therefore AB is perp. to LM.

Cor. *If a right line* AB *be perpendicular to ſeveral lines meeting in* B, *as* IB, LB, GB; *theſe lines are all in one plane.*

FIG. 140. For if any of them was out of the plane, AB would make an angle with it, greater or leſſer than a right angle.

PROP. V.

141. *Two right lines,* AB, CD, *perpendicular to a plane, are parallel.*

Make BDI a right angle, and DI = AB, and draw BI, AI, AD; AB is ⊥ to BD (Def. 5); and ∠ ABD = BDI; therefore the triangles ABD, DBI are equal (6. II), and AD = BI. Then the triangles ADI and ABI are equal (8. II); and ABI = ADI = a right angle. Therefore ID is ⊥ to DC, DA, DB; and therefore all three are in one plane (Cor. 4). Therefore AB, CD are in the ſame plane (Cor. Prop. II), and are likewiſe parallel (Cor. 3. 4. I).

Cor. 1. *Two parallel lines* AB, CD, *are in the ſame plane.*

Cor. 2. *A line drawn from one parallel to another, is in the ſame plane with them.*

By Cor. 1. and Cor. to Prop. I.

Cor. 3. *Through one point, there can be drawn but one line perpendicular to a plane.*

PROP. VI.

141. *If one* AB, *of two parallels, be perpendicular to a plane; the other will alſo be perpendicular to it.*

Suppoſe as in the laſt Prop. then the angles IDA, IDB, are right. Therefore DI is ⊥ to the plane ADB, in which AB, CD are (Cor. 1. 5);

therefore ID is ⊥ to CD; but CDB = a right angle. FIG.
Therefore CD is ⊥ to the plane EG. 141.

PROP. VII.

If FI *be perpendicular to the plane* DE, *and* FC *perpendicular to a line* AB, *drawn in that plane; then the line* CI *joining their ſections, is alſo perpendicular to the line* AB. 142.

For firſt, ſuppoſe CB ⊥ to CI, draw IG parallel to CB, then IG being ⊥ to CI and FI, is ⊥ to the plane CFI (4); and ACB is alſo ⊥ to the plane FCI (6); therefore BC is ⊥ to CI and to CF. And on the contrary being ⊥ to CF, it is alſo ⊥ to CI; otherwiſe it could not be ⊥ to the plane FCI; nor its parallel GI.

PROP. VIII.

Right lines AH, CI, *parallel to the ſame right line* EG, *though not in the ſame plane, are parallel to one another.* 143.

In the plane of the parallels AH, EG, let HG be ⊥ to EG. Alſo in the plane of the parallels EG, CI, draw GI ⊥ to EG. Therefore EG is ⊥ to the plane HGI; therefore AH, CI are alſo ⊥ to the ſame plane HGI (6), whence AH and CI are parallels (5).

PROP. IX.

If two planes AB, CD *be perpendicular to one another; and from any point* P *in one, a perpendicular* PN, *be let fall to the other; it ſhall fall upon the common ſection* PI. 144.

For the line PN ⊥ to the common ſection, is ⊥ to the plane AB (Def. 7), and if another perp. could be drawn which falls not upon the common

ſection;

FIG. ſection; then two perpendiculars might be let fall
144. from one point, which is abſurd (Cor. 3. 5).

Cor. *A line* NP *in one plane, perpendicular to the common ſection of two perp. planes, will be perp. to the other plane.*

PROP. X.

145. *Thoſe planes* AC, EG, *are parallel, when the ſame right line* IK, *is perpendicular to both.*

Draw DL parallel to IK, and draw ID, KL; then ſince the angles LKI = KID (hyp.) = IDL (6) = a right angle, therefore KLD is a right angle (16. III); therefore ID is parallel to KL (Cor. 3. 4. I); whence IKLD is a parallelogram, and IK = DL, therefore AC is parallel to EG (Def. 10).

Cor. *If a right line is perpendicular to one of two parallel planes, it is perpendicular to the other.*

PROP. XI.

145. *If two parallel planes* AC, EG, *be cut by a third* IL; *their common ſections are parallel*; ID, *and* KL.

For it was proved in the laſt prop. that IDLK is a parallelogram, and that ID and KL are parallel.

Or thus.

Let the plane IBED cut the parallel planes AC, EG, in the ſections ID, BE. Now if ID, BE be not parallel, or equidiſtant, they will meet ſome way; and conſequently the planes wherein they are placed, muſt meet, which is abſurd.

Cor. *If a line* ID *be parallel to the plane* EG; *all planes drawn through this line* ID, *ſhall interſect the plane* EG *in lines parallel to* ID, *and to one another.*

For KL is parallel to ID, and BE is parallel to ID, and therefore KL, BE are parallel to one another (8). FIG. 145.

PROP. XII.

Right lines AQ, BR, *cut by parallel planes*, G, H, I, *are cut proportionally*; AC : CE : : BD : DF. 146.

Draw AB, EF; and BE to cut the plane H in P. Then in the planes, BEF, EAB, the ſections PD, EF, as alſo CP, AB, will be parallel (11); therefore in the triangles BEF, EAB; AC : CE : : BP : PE : : BD : DF (12. II).

Cor. *The ſegments of parallel lines, cut off by parallel planes, are equal.*

PROP. XIII.

If two lines AB, AC, *cuting one another, be parallel to two other right lines*, ED, DF, *cuting one another, though not in the ſame plane; theſe lines will make equal angles*; BAC = EAD. 147.

Let AB = DE, AC = DF, and draw BE, AD, CF, and alſo BC, EF. Since AB, DE are parallel and equal, therefore BE, AD are equal and parallel (Cor. 3. 5. I). For the ſame reaſon CF, AD, are equal and parallel. Therefore BE, FC are parallel and equal (Prop. VIII. and Ax. 1). Therefore BC is equal and parallel to EF (Cor. 3. 5. I). The triangles BAC, EDF, have all their ſides equal, therefore ∠BAC = EDF (8. II).

PROP. XIV.

If two lines AB, AC, *which meet one another, be parallel to two other lines* DE, DF, *that alſo meet one another, though not in the ſame plane; the planes* BC, EF, *drawn through them, will be parallel.* 148.

Let

FIG. 148. Let AG be perpendicular to the plane EF, and GH, GI parallel to DE, DF; then GH, GI will be parallel to AB, AC. And ſince IGA, HGA are right angles, CAG, BAG, will be right angles (4. I); therefore GA is ⊥ to the plane BC, and ſince it is ⊥ to the plane EF (conſtruct.), therefore the planes BC, EF are parallel (10).

PROP. XV.

149. *If two planes* AB, CD, *which cut one another, be both of them perpendicular to a third plane* GH; *their common ſection* EF, *ſhall alſo be perpendicular to the third plane,* GH.

For a perpendicular to the plane GH, at the point F (in the common ſection of the planes AB, GH), muſt be ſomewhere in the plane AB (Def. 7). Alſo a perpendicular at F (in the common ſection of the planes CD, GH), muſt be ſomewhere in the plane CD (ibid.); therefore it muſt be in their common ſection; that is, the common ſection EF is ⊥ to the plane GH.

Cor. *The common ſection* EF *will be perpendicular to* FD, *or* FB, *the ſection of either plane with the third.*

PROP. XVI.

150. *In a ſolid angle* A, *contained under three plane ones,* BAD, DAC, BAC; *any two of them is greater than the third.*

Let BAC be the greateſt, and let ∠BAE = BAD, and AD = AE. And draw BEC, BD, DC. The triangle BAE = BAD, for BA, AE are equal to BA, AD, and ∠BAE = BAD, therefore BE = BD, and AE = AD (6. II). But BD + DC is greater than BC (5. II), and DC greater than EC. And ſince AD = AE, and AC

common, ∠CAD is greater than CAE (Cor. 6. II). FIG.
Therefore BAD + CAD is greater than BAC. 150.

PROP. XVII.

Every ſolid angle is contained under leſs plane angles 151.
than four right angles.

Suppoſe a plane to cut the ſides of the angle, and to make a polygon BCDE, to conſiſt of as many triangles, as there are to make up the ſolid angle A.

Let X = ſum of all the external angles of the polygon B, C, D, *&c.* Y = ſum of all the angles at the baſes of the triangles compoſing the ſolid angle, EBA, ABC, *&c.* Then will X + 4 right angles = Y + A (2. II). But ſince EBA + ABC is greater than B (16), *&c.* therefore Y is greater than X, and conſequently A is leſs than 4 right angles.

PROP. XVIII.

Theſe ſolid angles are equal A, G; *which are con-* 151.
tained under the ſame number of plane angles, alike 152.
ſituated, and having the ſame inclinations, reſpectively.

For ſince ∠BAC = HGI; CAD = IGK, *&c.* therefore if HGI be laid upon BAC, they will coincide, and GI will fall upon AC. Alſo if IGK be laid upon CAD, they will likewiſe coincide. And moreover, ſince the inclination of the planes HGI and KGI is the ſame as BAC and DAC; therefore the ſolid under HGIK will exactly coincide with that under BACD. For the ſame reaſon the ſolid, under the planes IGKL and CADE, will likewiſe concide; and alſo the ſolid under KGLH and DAEB will coincide; and thoſe under LGHI, and EABC, will coincide; and ſo the whole ſolid angle G will coincide with the whole ſolid angle A, and conſequently they are equal (Ax. 8).

PROP. XIX.

FIG. 153. 154. *If two ſolid angles* A, B, *be contained under three plane angles reſpectively equal, and alike ſituated; the like planes have the ſame inclination to one another.*

Let ∠KAD = MBG, KAE = MBH, and EAD = HBG; the ∠ made by KAD and KAL, will be equal to that made by MBG and MBN. For make BM = AK, and let KD, KL be ⊥ to AK, and MG, MN ⊥ to BM. Draw LD, NG; in the triangles KAD, MBG, ∠ KAD = MBG, and K, M right, and AK = BM; therefore KD = MG, and AD = BG (7. II). For the ſame reaſon, in the triangles KAL, MBN; KL = MN, and AL = BN. And in the triangles LAD, NBG; LA, AD are equal to NB, BG, and ∠A = B, therefore LD = NG (6. II). In the triangles KLD, MNG; the three ſides are equal; therefore ∠DKL = ∠GMN, which are the inclination of the planes. And the ſame way it is demonſtrated for the other planes.

Cor. *Theſe ſolid angles are equal, which are contained under three plane angles, reſpectively equal.*

For the planes of theſe angles will have the ſame inclination to one another reſpectively (19); and conſequently the ſolid angles, contained thereby, will be equal (18).

SCHOLIUM.

It is evident from hence, that a ſolid angle, conſiſting of 3 planes, is determined from the quantity of the 3 plane angles it conſiſts of. For (fig. 153), 153. the triangle KLD, which is its baſe, is determined from the three ſides, KL, LD, KD, being given. And if the point A be alſo given; the planes AKL, ALD, AKD, are capable of no alteration in their poſition; and ſo the ſolid angle A is determined. But although a ſolid angle of 3 plane angles is determined from the quantity of the angles alone; yet when 4 or more planes are concerned, the quantity of their angles is not ſufficient. This will be plain by inſpecting fig. 155. Where the baſe of the ſolid angle A, is the trapezium BCDI. For the 4 ſides of the trapezium alone are not ſufficient to determine its figure; and by altering its figure, the poſition of the planes is altered (though the ſeveral angles are not), and conſequently the quantity of the ſolid angle A, is altered. So that the ſolid angle can no more be determined, from the plane angles given; than a trapezium can, by having all its ſides given; and much leſs can it be ſo in polygonal angles and baſes. 155.

PROP. XX.

If there be two ſolid angles A, G, *and the ſides of one,* AB, AC, AD, AE, *be reſpectively parallel to the ſides* GH, GI, GK, GL, *of the other; theſe ſolid angles will be equal.* 151. 152.

For ſince AB, AC, are parallel to GH, GI; $\angle BAC = HGI$ (13); for the ſame reaſon $\angle CAD = IGK$, $DAE = KGL$, $EAB = LGH$. Moreover, as AB, AD are parallel to GH, GK; $\angle BAD = HGK$, therefore the ſolid angle made by the three planes BAC, CAD, BAD, is equal to that made by the three planes HGI, IGK and HGK (Cor. 19). For the ſame reaſon the ſolid angle made by the three planes CAD, DAE, CAE is equal to that made by IGK, KGL, IGL. And for the ſame reaſon the ſolid angle A made by DAE, EAB = ſolid angle G made by KGL, LGH. And ſolid angle made by EAB, BAC = ſolid angle made by LGH, HGI. Whence all the parts of the ſolid angles A, G, being mutually equal, and having a like

FIG. 151. 152. like ſituation; the whole angle A, muſt be equal to the whole angle B.

Cor. *In two ſolid angles* A, G, *whoſe planes* BAC, CAD, *&c. are reſpectively parallel to the planes* HGI, IGK, *&c. theſe ſolid angles will be equal.*

For it comes to the ſame thing, whether the lines AB, GH, be parallel, or the planes BAC, HGI, *&c.* (14).

BOOK VI.

Of Solids.

DEFINITIONS.

FIG. 155. 1. A *Pyramid*, is a ſolid ABD, made by the motion of a line as AB, along the circumference BCDIB of the plane figure BD, the other end at A, remaining fixt. The figure BCDA is called the *baſe* of the pyramid. The fixt point A is the *vertex*. If the baſe be a triangle, it is a *triangular pyramid*; if a polygon, a *multangular pyramid*. 150.

156. 2. A *cone* is a ſolid generated by a line AB moving about the circle BCD, the end A remaining fixt. The *vertex* is the fixt point A. The *axis* is the line AO drawn from the vertex to the center O of the circle. The *baſe* is the circle BCD. The *ſide* is AB or AD. It is called a *right cone*, if the axis is perpendicular to the baſe; otherwiſe an *oblique* or *ſcalene cone*. An *equilateral cone*, is a right cone whoſe ſide is equal to the diameter of the baſe.

157. 3. A *cylinder* is a ſolid, formed by a line FB moving about two equal and parallel circles, ſo as that the moving line always keep parallel to the line PO joining their centers. The circle FG or BD is called the *baſe*. The line PO, drawn between the centers of the circles, is the *axis*. If the axis is perpendicular to the baſe, it is a *right cylinder*; if not, an *oblique one*. FB or GD is the *ſide*. If the ſide of a right cylinder be equal to the diameter of the baſe, it is called an *equilateral cylinder*.

FIG. 4. A *priſm* is a ſolid, as ACEH, whoſe ends are
159. two ſimilar equal plane figures, and parallel to one
another; and the ſides, are parallelograms. The
baſe is the plane figure at either end ABCD or
HGEF. If all the ſides are perpendicular to the
baſe, it is a *right priſm*; otherwiſe an *oblique one*.
158. If the baſe is a triangle, it is a *triangular priſm*; if
a polygon, a *multangular priſm*.

Cor. *A cylinder is a priſm of an infinite number of ſides.*

160. 5. A *parallelopipedon* is a priſm contained under ſix plane figures, whoſe baſes, and oppoſite ſides are parallel, as ABD. If the ſides are all perpendicular to the baſes, it is an *upright parallelopipedon*; if not, an *oblique one*.

161. 6. A *cube* is a ſolid contained under ſix equal ſquares, ſet perpendicular to one another, as AB.

7. A *polyedron*, is a ſolid contained under ſeveral rectilineal figures.

8. A *regular ſolid* or *body*, is a ſolid contained under ſome number of equal and regular plane figures of the ſame ſort; otherwiſe, they are *irregular bodies*.

160. 9. *Hight*, of a ſolid, is the perpendicular falling from the vertex or top, upon the baſe, as BP.

10. *Fruſtum*, of a ſolid, is the lower part, cut off by a plane parallel to the baſe.

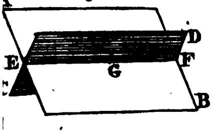
11. *Similar pyramids*, are thoſe contained under ſimilar plane figures, equal in number, and alike placed.

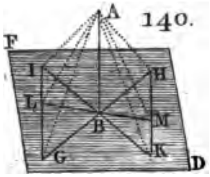
12. *Similar ſolids* are thoſe which are made up of an equal number of ſimilar pyramids, alike placed: or which may be reſolved into ſuch.

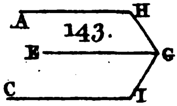
13. *Area*, is the quantity of the ſuperficies of any plane figure.

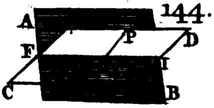
14. *Bodies* are ſaid to *touch* one another, when they meet, but do not cut or enter into one another.

139.









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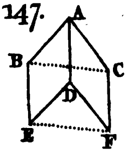

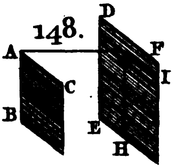

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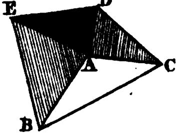

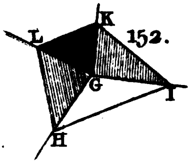

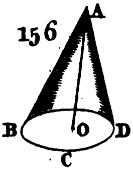

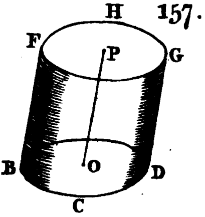

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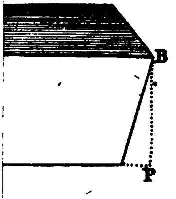

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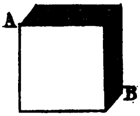

PROP. I.

FIG.

In any parallelopipedon EH, *the oppoſite planes* AE, HB, *are ſimilar and equal parallelograms.* 162.

The plane AC, cuting the parallel planes AG, DB, make the ſections AH, DC parallel (11. V). And the ſame plane AC, cuting the parallel planes AE, HB, make the ſections AD, HC parallels (ibid.); therefore ADCH is a parallelogram. By the ſame reaſoning, all the other planes are parallelograms. Therefore BG = CH = DA = EF (1. III). And ſince DA, AF are parallel to CH, HG; therefore ∠DAF = CHG (13. V); therefore the parallelogram AE = HB (Ax. 8), having equal ſides and angles. And the ſame way it is ſhewn of the other oppoſite planes.

PROP. II.

If a priſm HC, *be cut by a plane parallel to the baſe* AC; *the ſection* EG, *will be ſimilar and equal to the baſe.* 163.

Since AE, BF, CG, DI are parallel (Def. 4), and the plane ABFE is cut by the parallel planes AC, EG, the ſections AB, EF will be parallel, therefore ABFE is a parallelogram, and EF = AB (1. III). For the ſame reaſon FG = BC; GI = CD, and EI = AD. And ſince AB, BC are parallel to EF, FG; ∠ABC = EFG (13. 5). After the ſame manner ∠C = G, and I = D, and E = A. Whence the figure EFGI is ſimilar and equal to ABCD (Ax. 8), having the ſides and angles all equal.

Cor. *If a parallelopipedon be cut by a plane parallel to any ſide; the ſection will be ſimilar and equal to that ſide.*

FIG. 163. For in that ſolid, any ſide may be taken for the baſe (1).

PROP. III.

The ſurface of any polyedron, is equal to the ſum of the areas of all the figures that incloſe it.

For all theſe figures make up the ſurface, therefore the ſum of their areas is equal to the area of the whole (Ax. 2).

Cor. 1. *The ſurface of a pyramid is equal to the ſum of all the triangles incloſing it, together with the baſe.*

159. Cor. 2. *The ſurface of an upright priſm* AE, *is equal to the rectangle of its hight,* CE, *and the circumference of its baſe,* GEFH.

For all the ſides are rectangles of the ſame hight, all which are equal to a rectangle, whoſe baſe is the ſum of all theſe, and hight the ſame (8. III).

Cor. 3. *The ſurface of any regular body is equal to the area of one of the faces, multiplied by the number of them.*

PROP. IV.

164. *The curve ſurface of a right cylinder* AD, *is equal to the rectangle of its hight, and the circumference of the baſe:* BD × CKDC.

Suppoſe FK, OI to be drawn upon its ſurface parallel to the axis, and extremely near together. Then the part of the ſurface OK is equal to the ſmall parallelogram OIKF, or OI × IK (Cor. 2. 6. III). In like manner the whole ſurface may be divided into ſuch parallelograms, the ſum of all which, will be = the ſum of all the IK's × OI; that is, the curve ſurface will be = the circumference CKDC × OI.

Cor.

FIG. 164. Cor. 1. *The curve ſurface of a right cylinder, is equal to a circle whoſe radius is a mean proportional between the ſide* AB, *and diameter of the baſe* BD.

For let R, C be the radius and circumference of this circle, A its area. Then AC : R :: R : CD (hyp.), and AC : ½R :: 2R : CD (Cor. 3. 12. Proportion) :: C : circumference CKDC (9. IV). Therefore AC × CKDC = ½RC; that is, the ſurface of the cylinder = A (34. IV).

Cor. 2. *As half the radius of the baſe : to the ſide :: ſo the baſe of the cylinder : to its curve ſurface.*

For the baſe = $\frac{CD \times CKDC}{4}$ (34. IV), and $\frac{CD \times CKDC}{4}$: ſurface AC × CKDC :: $\frac{CD}{4}$: AC (Cor. 1. 5. Proportion).

Cor. 3. *The curve ſurfaces of right cylinders, are in the complicate ratio of the hights, and diameters of the baſes.*

For their equal rectangles are in that ratio (Cor. 2. 8. III), and the diameters are as the circumferences (9. IV).

PROP. V.

165. *The curve ſurface of a right cone, is equal to the area of a triangle, whoſe hight is the ſide* AB, *and baſe the circumference of the cone's baſe*, BKDB.

Take the very ſmall arch IK, and draw AI, AK. Then the part of the ſurface AIK coincides with the ſmall iſoſceles triangle AIK, whoſe baſe is IK, and hight AI. In like manner the whole curve ſurface of the cone, may be ſuppoſed to conſiſt of ſuch triangles, whoſe common hight is AI, and baſes ſo many KI's. All which triangles are equal to the triangle whoſe

FIG. 166. : PH : : PB : PA $= \frac{PH \times PB}{BC - PH}$ (13, Proportion); but ſurface of the cone ABD $= RC = nRR = n \times AB \times BC$ (Cor. 2. 5) $= n \times \overline{AP + PB} \times BC = n \times \overline{\frac{PH \times PB}{BC - PH} + PB} \times BC = n \times \frac{PH \times PB + BC \times PB - PH \times PB}{BC - PH} \times BC = n \times \frac{BC \times PB}{BC - PH} \times BC$.

Alſo ſurface of the cone APE $= rc = nrr = n \times AP \times PH$ (Cor. 2. 5) $= n \times \frac{PH \times PB \times PH}{BC - PH}$. Therefore their difference, or the ſurface of the fruſtum is $n \times \frac{BC^2 \times PB}{BC - PH} - n \times \frac{PH^2 \times PB}{BC - PH} = n \times PB \times \frac{BC^2 - PH^2}{BC - PH} = n \times PB \times \overline{BC + PH}$ (12. I) = the circle whoſe radius is $\sqrt{PB \times \overline{BC + PH}}$, and circumference $n \times \sqrt{PB \times \overline{BC + PH}}$.

PROP. VII.

167. *The ſurfaces of ſimilar ſolids* AD, PS, *are as the ſquares of their homologous ſides,* AB^2 *and* PQ^2.

Draw the diagonals AC, PR. Then ſince the bodies are reſolvable into ſimilar pyramids (Def. 12), which are contained under ſimilar plane figures (Def. 11). Let the planes incloſing them, be ABC, PQR, and AGC, PIR, *&c.* which being ſimilar, it is AB : PQ : : AC : PR : : AG : PI : : GE : IT, *&c.* (13. II); and ſince $AB^2 : PQ^2$: : triangle ABC : PQR (18. II), and $AC^2 : PR^2$: : ACG : PRI; and $AG^2 : PI^2$: : trapezium AE : PT (20. III); and $GE^2 : IT^2$: : GD : IS, *&c.* therefore $AB^2 : PQ^2$: : ABC : PQR : : ACG : PRI : : AE : PT : : GD : IS *&c.* whence $AB^2 : PQ^2$: : BG + AE + GD *&c.* : QI + PI + IS *&c.* (10. Proportion) : : ſurface of AD : ſurface of PS.

Cor. *Similar parts of the ſurfaces of ſimilar ſolids, are as the ſquares of the homologous ſides.* FIG. 167.

PROP. VIII.

168. *A right triangular priſm* ABCFHE *is equal to an oblique one* APIGHD, *of the ſame length* AH, *contained within the ſame three parallel lines* EP, HA, FI; *or the planes paſſing through them.*

For AH = PD = BE = IG = CF (1. III), whence PB = DE, IC = GF, and AP, AB, AC, AI being parallel to HD, HE, HF, HG (Cor. 3. 5. I), the ſolid angle A = ſolid angle H (20. V). For the ſame reaſon the ſolid angles at P, B, C, I, are reſpectively equal to thoſe at D, E, F, G. And ſince the ſides are all equal, each to each, therefore the two ſolids APBCI and HDEFG will exactly coincide, and be equal the one to the other (Ax. 8); and therefore the rectangled priſm HEFCAB = the oblique one HDGIAP.

Cor. 1. *If a parallelopipedon* AB, *be cut by a plane paſſing through the diagonals of the oppoſite planes; it ſhall be cut into two equal parts.* 169.

For the triangle CGF = CBF, and DAE = DHE (1. III); and the length AG = length BH; therefore if the two priſms CFA, and CFH be laid ſo, that H may coincide with A, and EHB with DAG, their planes will concide, and each of them being oblique, is equal to a right one of the ſame length (8).

Cor. 2. *Hence any priſmatic ſolid cut obliquely by parallel planes, is equal to the ſame cut off at right angles, and of the ſame length.*

For any ſuch ſolid may be divided into triangular priſms, by planes paſſing through both ends of the

ſolid.

FIG. ſolid. And each triangular priſm cut obliquely, 169. is equal to one of the ſame length, cut at right angles (8).

PROP. IX.

170. *If a parallelopipedon* AS, *be cut by a plane, paſſing through* O *the middle of the diameter* CQ; *the plane biſects it.*

Let the diagonals AD, BC cut each other in F; and RQ, PS, in I. Draw the axis FI, which cuts CQ in O, becauſe BCRQ is a parallelogram (2. and Cor. 3. III); and FO = OI. Let the plane EHOVX be parallel to ABDC. Then the parallelopip. AX = half AS. Let any plane GTOLN paſs through O. And let the ſolid be cut by the two planes ADSP, and CBQR, into four triangular priſms.

The two oppoſite ſolids OTGEH and OLNXV, are equal; for the ſides are parallel (11. V), and equal (Cor. 3. III). And therefore the ſolid angles, at the correſpondent points, are equal (10. V); therefore the ſolid EOG = XON. Therefore in the oppoſite priſms ACI, and BDI, the ſolids contained between the planes EVXH and GTLN, are equal. And it is proved the ſame way, that the ſolids, in the oppoſite priſms ABI, and DCI, contained between the planes EVXH and GLNT, are equal. And therefore ſince AX is half the parallelop. the plane GTNL cuts off half the parallelop. or divides it into two equal parts.

Cor. *The axis* FI, *and diagonal* CQ, *biſect each other.*

For they are both in the parallelogram BCRQ (Cor, 3. III).

PROP. X.

FIG. 171.

Parallelopipedons upon the ſame baſe CDFI, *and between the ſame parallel planes,* CIFD *and* BHVQLA, *are equal.*

The triangles LAI and KEF are equal and ſimilar (6. II); and the priſm KEFDQH = LAICBG; ſubtract the common ſolid E*r*LQ*s*G, and the ſolid AI*r*EBC*s*Q = L*r*FKG*s*DH; add the priſm I*r*FC*s*D, and the ſolid paral. CDFIQEAB = CDFIHKLG upon the ſame baſe ID.

Again, the triangles FVK and DMH are ſimilar and equal (6. II), and the priſm FVKIOL = DMHCLG; ſubtract the common priſm M*t*KP*n*L, and the ſolid FVM*t*IOP*n* = D*t*KHC*n*LG; add the priſm DF*t*CI*n*, and the ſolid par. FVMDIOPC = DFRHCILG = CDFIQEAB.

PROP. XI.

172.

Parallelopipedons of equal baſes and hights are equal.

Let the parallelogram AGIC be the baſe of the parallelopiped. Draw BH, DF parallel to AG, AC. The ſolid pip. upon the baſe AGI = that on the baſe ACI (Cor. 1. 8); and ſolids on ABE and EFI, are = thoſe on ADE, and EHI (ibid.). Take the two laſt from the firſt, and there remains the ſolid on DH = ſolid on BF. But parallelogram DH = BF (4. III). Therefore ſolid pips. on equal baſes and hights are equal, when the angle at E is the ſame. Moreover, the pip. on the baſe BCEF is equal to that on the baſe EOPF, and the ſame hight (Cor. 2. 8); reckoning OP or EF the length of the ſolid: Whence the parallelopip. on the baſe DH, is equal to the pip. on the baſe EP, and hight the ſame.

PROP.

FIG.

PROP. XII.

173. *Parallelopipedons of the ſame hight are in proportion as their baſes.*

Let BN be the baſe of a parlopip. divide the baſe into any number of equal parts at D, E, F, G, *&c* and draw planes ‖ to ABC; then the parlopips. ſtanding upon CD, DE, EF, *&c.* will be all equal (11); whence the pip. AK is as multiple of AD, as the baſe BK is of the baſe BD, alſo the pip. LN is as multiple of AD, as the baſe ON is of BD. Whence it will be as pip. AK : pip. LN : : baſe BK : baſe ON (Def. 4. Proportion). Moreover, let the baſe PQ be = ON, and hight QR = AB, then the pip. PR = LN (11), whence pip. AK : pip. PR : : baſe BK : baſe PQ.

Cor. 1. *Parallelopipedons of equal baſes are as their hights.*

For in rectangled ones, any ſide or face may be taken for the baſe; and rightangled ones are equal to oblique ones, between the ſame parallel planes (10).

Cor. 2. *Parallelopipedons are to one another, in the complicate ratio of their baſes and hights.*

PROP. XIII.

175. *If two parallelopipedons,* AD, FI, *be equal; their baſes and hights are reciprocally proportional;* AC : FH : : HI : CD.

Suppoſe the ſides CD, HI perpendicular to the baſes, and make HM = CD. Then baſe AC : baſe FH : : ſolid AD or FI : ſolid FM (12) : : HI : HM or CD (Cor. 1, 12). And if the pips. be oblique, inſtead of ſuppoſing CD, HI to be the ſides,

let them be the hights, and then oblique pips. FIG.
being equal to upright ones (10); the proportion 175.
continues the ſame.

Cor. *If the baſes and hights of two parallelopipedons be reciprocally proportional, they are equal.*

For ſince baſe AC : baſe FH : : HI : CD (hyp.), therefore $AC \times CD = FH \times HI$ (12. Proportion), and ſolid BE : ſolid FI : : $AC \times CD : FH \times HI$ (Cor. 2. 12). Therefore ſolid BE = ſolid FI (Ax. 7. Proportion).

PROP. XIV.

All priſms whatſoever, ABD, PSR, *of equal baſes* 174.
and hights, are equal.

For any polygonal baſe BD may be divided into triangles, by diagonal lines; and the polygonal priſm may likewiſe be divided into triangular priſms, by planes paſſing through theſe diagonals; each of which triangular priſms is equal to half a parallelopipedon ſtanding on double the baſe (9); and as all theſe triangular priſms make up the polygonal priſm, this priſm muſt be equal to a parallelopip. of the ſame baſe and hight, and that equal to the priſm PRS of an equal baſe and hight (Cor. 1. 12).

Cor. 1. *Priſms of equal baſes are as their hights; and of equal hights, are as their baſes.*

For they may be divided into triangular priſms, which are half of parlopips. on double the baſe, and theſe pips. are as their hights, when the baſe is the ſame; or as the baſes, when the hight is the ſame. (Cor. 2. 12).

Cor. 2. *All priſms are to one another in the complicate ratio of their baſes and hights.*

Cor.

FIG. 174. Cor. 3. *Bodies of equal ſurfaces may be very different in ſolidity. And equal ſolids may have ſurfaces vaſtly different.*

Cor. 4. *In equal priſms, the baſes and hights, are reciprocally proportional; and the contrary.*

PROP. XV.

The ſolidity of any priſm is equal to the product of the baſe and hight.

For a priſm is equal to a right-angled parallelopip. of the ſame baſe and hight; and that is equal to the product of its baſe and hight; or (which is the ſame) it is equal to the ſolid ſpace contained under the planes of the upright parallelopipedon (Def. 5).

PROP. XVI.

176. *Equiangular parallelopipedons* AB, CD, *are in the complicate ratio of their homologous ſides,* FG, GI, GB, *and* OE, EH, ED.

Let FP, OK be ⊥ upon the baſes IB, HD. Then by reaſon of the equal angles at G and E, the triangles GFP, EOK will be ſimilar; and FP : OK :: FG : OE (13. II). The parallelograms IB and HD being equiangular at G and E, are to one another as IG × GB, to HE × ED (10. III). The parlepip. AB : CD :: baſe IB × FP : baſe HD × OK (Cor. 2. 12) :: IG × GB × FP : HE × ED × OK :: GI × GB × GF : HE × ED × EO (7. Proportion).

PROP. XVII.

177. *Pyramids upon the ſame baſe, and of equal altitudes, are equal:* ACF = HCF.

Draw the plane AH, through the tops of the pyramids, which will be parallel to CF. Alſo through any points of the pyramids, draw the plane BE, alſo parallel to CF; then by ſimilar triangles, CF : BD :: AC : AB (13. II) :: HC : HL (12. II) :: CF : LE (13. II); therefore BD = LE. And by the ſame reaſoning, BO = LI; and DO = EI. Whence the ſection BOD = LIE (8. II). Therefore if another plane NP be drawn very near, and parallel to BE, the ſegments of the pyramids, ND, PL, comprehended between theſe planes, will be equal (14). And therefore if never ſo many ſuch planes be drawn, the parts intercepted will always be equal. Therefore the ſum of all the parts of one pyramid, will be equal to the ſum of all the parts of the other; or the pyramid ACGF = pyramid HCGF (Ax. 2). FIG. 177.

Cor. 1. *If a pyramid is cut by a plane parallel to the baſe, the ſection will be ſimilar to the baſe.*

For by ſimilar triangles, it is AC : AB :: CG : BO :: GF : OD :: CF : BD.

Cor. 2. *If a cone be cut by a plane parallel to the baſe; the ſection will be a circle.*

For a cone may be conſidered as a pyramid of an infinite number of ſides.

PROP. XVIII.

Every priſm is three times the pyramid of the ſame baſe and hight. 178.

Let AFC be a triangular priſm, draw AC, CF, FD, the diagonals of the three parallelograms. The triangle ACB = ACD (1. III); therefore pyramid ACBF = ACDF, their vertexes being in F (17); likewiſe triangle DFA = DFE (1. III), and pyramid DFAC = DFEC, their vertexes being

FIG. 178. ing in C (17). But ACDF and DFAC are one and the ſame pyramid. Therefore the three pyramids, that make up the priſm, are equal to one another, ACBF = ACDF = DFEC; and each of them is $\frac{1}{3}$ the priſm.

And ſince any polygonal priſm may be reſolved into triangular ones; and the pyramid, upon the ſame baſe, into triangular pyramids. Then all the triangular priſms will be triple to all the triangular pyramids; and conſequently the whole priſm triple to the whole pyramid.

Cor. 1. *Pyramids of the ſame hight, are to one another as their baſes.*

For priſms, which are triple of them, are in that ratio (Cor. 1. 14). Whence,

Cor. 2. *Pyramids of the ſame or equal baſes are as the hights.*

Cor. 3. *Pyramids are to one another in the complicate ratio of their baſes and hights.*

Cor. 4. *Pyramids of equal baſes and hights are equal.*

Cor. 5. *In equal pyramids the baſes and hights are reciprocally proportional; and the contrary.*

For priſms are in that ratio (14. and Cor.).

PROP. XIX.

Cylinders of equal baſes and hights are equal.

For cylinders are nothing but priſms, whoſe baſes are polygons of an infinite number of ſides. And theſe priſms are equal (14).

Cor. 1. *Cylinders of equal baſes are as the hights.*

Cor. 2. *Cylinders of equal hights are as the baſes.*

Cor. 3. *Cylinders are to one another in the complicate ratio of their baſes and hights.* FIG. 178.

Cor. 4. *In equal cylinders, the baſes and hights are reciprocally proportional: and the contrary.*

All this follows from Prop. 13, 14, and Corol.

PROP. XX.

Every cone is the third part of a cylinder of the ſame baſe and hight.

For cones and cylinders may be conſidered as pyramids, and priſms, whoſe baſes are regular polygons of an infinite number of ſides. And conſequently the cone = $\frac{1}{3}$ the cylinder (18).

Cor. 1. *Cones of equal baſes, are as their hights.*

Cor. 2. *Cones of equal altitudes, are as the baſes.*

Cor. 3. *Cones are to one another in the complicate ratio of the baſes and hights.*

Cor. 4. *In equal cones, the baſes and hights are reciprocally proportional.*

All theſe things appear by Prop. 13 and 14, and 19. For the cylinders are in that ratio, and the cone is $\frac{1}{3}$ the cylinder.

PROP. XXI.

The fruſtum of a pyramid or cone BG, *is equal to the third part of a parallelopipedon, of the ſame hight, and its baſe equal to the ſum of the baſes of the fruſtum* BOD + EFG, *together with a mean proportional between theſe baſes.* 179.

Draw EB, GD to meet in A, the top of the whole ſolid, and let ACP be ⊥ to the baſe. Draw the diameters BD, EG; then the two baſes BOD,

FIG. 179. EFG will be ſimilar (Cor. 1, 2. 17). Whence, baſe BOD : baſe EFG :: $\mathrm{BD}^2 : \mathrm{EG}^2$ (20. III). Therefore ſuppoſe $\frac{\text{baſe BOD}}{\mathrm{BD}^2} = \frac{\text{baſe EFG}}{\mathrm{EG}^2} = n$, or baſe BOD $= n \times \mathrm{BD}^2$, and baſe EFG $= n \times \mathrm{EG}^2$. By ſimilar triangles, EG : BD :: (AE : AB ::) AP : AC (13. II), and EG — BD : BD :: CP : AC $= \frac{\mathrm{BD} \times \mathrm{CP}}{\mathrm{EG} - \mathrm{BD}}$. Then the whole pyramid or cone $=$ baſe EFG $\times \frac{1}{3}$AP (18, 20) $= \frac{n \times \mathrm{EG}^2}{3} \times \overline{\mathrm{CP} + \mathrm{AC}} = \frac{n \times \mathrm{EG}^2}{3} \times \mathrm{CP} + \frac{n \times \mathrm{EG}^2}{3} \times \frac{\mathrm{BD} \times \mathrm{CP}}{\mathrm{EG} - \mathrm{BD}}$ $= \frac{n \times \mathrm{EG}^3 \times \mathrm{CP} - n \times \mathrm{EG}^2 \times \mathrm{BD} \times \mathrm{CP} + n \times \mathrm{EG}^2 \times \mathrm{BD} \times \mathrm{CP}}{3 \times \overline{\mathrm{EG} - \mathrm{BD}}}$ $= \frac{n \times \mathrm{EG}^3 \times \mathrm{CP}}{3 \times \overline{\mathrm{EG} - \mathrm{BD}}}$. And the top part ABD $= \frac{\text{baſe BOD}}{3} \times$ AC (18, 20) $= \frac{n \times \mathrm{BD}^3 \times \mathrm{CP}}{3 \times \overline{\mathrm{EG} - \mathrm{BD}}}$; this taken from the whole, leaves $\frac{n \times \mathrm{CP}}{3} \times \frac{\mathrm{EG}^3 - \mathrm{BD}^3}{\mathrm{EG} - \mathrm{BD}}$ for the fruſtum $= \frac{\mathrm{CP}}{3} \times \overline{n \times \mathrm{EG}^2 + n \times \mathrm{EG} \times \mathrm{BD} + n \times \mathrm{BD}^2}$, becauſe $\overline{\mathrm{EG}^2 + \mathrm{EG} \times \mathrm{BD} + \mathrm{BD}^2} \times \overline{\mathrm{EG} - \mathrm{BD}} = \mathrm{EG}^3 - \mathrm{BD}^3$ (Cor. 1. 8. I), and $n \times \mathrm{EG}^2 =$ baſe EFG, $n \times \mathrm{BD}^2 =$ baſe BOD, and $n \times \mathrm{EG} \times \mathrm{BD}$ is a mean between them (Cor. 2. 12. Proportion).

Cor. *If* $n = \frac{\textit{baſe}\ \mathrm{EFG}}{\mathrm{EG}^2}$, *the fruſtum* $= \frac{n \times \mathrm{CP}}{3} \times \frac{\mathrm{EG}^3 - \mathrm{BD}^3}{\mathrm{EG} - \mathrm{BD}}$.

PROP. XXII.

180. *In ſimilar ſolids*, AD, PS, *the homologous ſides are*
181. *proportional*; AB : AF :: PQ : PV.

Through the diagonals AC, FG, GD, and PR, FIG.
VI, IS, let planes be drawn to divide the ſolids 180.
into pyramids. Then ſince theſe pyramids are ſi- 181.
milar (Def. 12), and their planes ſimilar figures (Def. 11); therefore if ABC, PQR, and ACG, PRI, and AGF, PIV, &c. be ſimilar planes belonging to the ſimilar pyramids; it will be AB : PQ (:: AC : PR :: AG : PI) :: AF : PV. Alſo AF : PV :: (FG : VI ::) FE : VT, &c.

Cor. *The like planes or ſurfaces, which incloſe ſimilar ſolids, are proportional.*

For ſince AB : PQ :: AF : PV; AB^2 : PQ^2 :: AF^2 : PV^2 (Cor. 3. 18. Proportion); that is, ABCG : PQRI :: AGEF : PITV (20. III).

PROP. XXIII.

Similar triangular pyramids ABCD, PQRS *are as* 182.
the cubes of their homologous ſides, AB^3 *and* PQ^3. 183.

Suppoſe CE, BF drawn parallel to AD, and RT, QV, ∥ to PS; and the planes DFE, SVT, ∥ to ABC, and PQR; and ſo the priſms AF, and PV, compleated.

Then ſince the pyramid ABCD = $\frac{1}{3}$ priſm, AF; and pyramid PQRS = $\frac{1}{3}$ priſm PV; therefore pyramid ABCD : pyramid PQRS :: priſm AF : priſm PV (5. Proportion) :: AB × AC × AD : PQ × PR × PS (16).

But AB : PQ :: AB : PQ,
AB : PQ :: AC : PR (22),
AB : PQ :: AD : PS (22).

Therefore AB^3 : PQ^3 :: AB × AC × AD : PQ × PR × PS (18. Proportion) :: pyramid ABCD : pyramid PQRS.

Cor. *Any ſimilar pyramids are as the cubes of the homologous ſides.*

FIG. 182. 183. For they may be divided into ſimilar triangular pyramids, all which are in that proportion, and their ſums in the ſame proportion (10. Proportion).

PROP. XXIV.

180. 181. *All ſimilar ſolids*, AD, PS, *are to one another, as the cubes of their homologous ſides*, AB, *and* PQ.

Let the planes AC, PQ, and FG, VI, and GD, IS, &c. divide the bodies into ſimilar pyramids. Then ſince AB : PQ :: AG : PI :: EG : TI, &c. (22). Therefore

$AB^3 : PQ^3 ::$ pyr. ABC : pyr. PQR (23),
and $AB^3 : PQ^3 :: AG^3 : PI^3 ::$ pyr. AGC : pyr. PIR :: pyr. AGF : pyr. PIV.
and $AB^3 : PQ^3 :: EG^3 : TI^3 ::$ pyr. FGE : pyr. VIT :: pyr. EGD : pyr. TIS, &c.
Therefore
$AB^3 : PQ^3 ::$ pyr. ABC + AGC + AGF + FGE + EGD, &c. : pyr. PQR + PIR + PIV + VIT + TIS, &c. :: ſolid AD : ſolid PS.

Cor. *If four lines* A, B, C, D *be in continual proportion; then as the firſt* A *to the fourth* D; *ſo any ſolid deſcribed on the firſt* A, *to a ſimilar one, on the ſecond* B.

For A : D :: $A^3 : B^3$ (23. Proportion) :: ſolid upon A : ſolid upon B (24).

PROP. XXV.

184. *If four lines be proportional,* AB : CD :: GH : LM; *ſimilar ſolids, alike deſcribed, upon two and two, ſhall alſo be proportional:* ABE : CDF :: GHK : LMN.

And if four figures be proportional, and two and two ſimilar; their homologous ſides ſhall be proportional.

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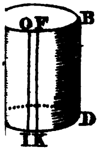
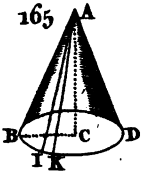


165



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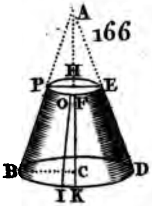

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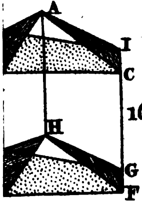

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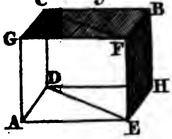

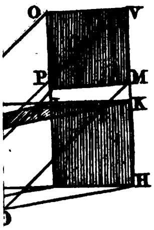

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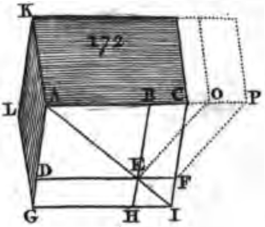

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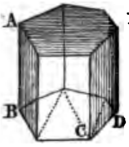

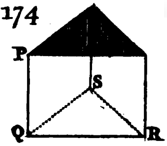

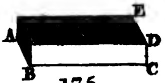

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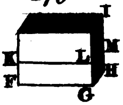

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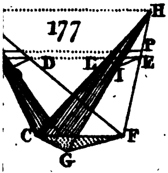

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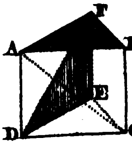

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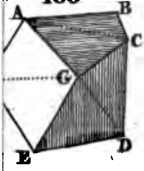

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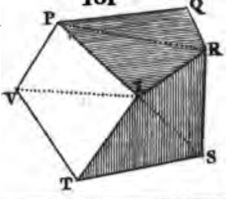


For ſince $AB : CD :: GH : LM$ (hyp.), FIG. 184.
therefore $AB^3 : CD^3 :: GH^3 : LM^3$ (Cor. 3. 18, Proportion),
whence $ABE : CDF :: GHK : LMN$ (24).
Again, if the ſolids be ſimilar,
and $ABE : CDF :: GHK : LMN$ (hyp.),
then $AB^3 : CD^3 :: GH^3 : LM^3$ (24),
whence $AB : CD :: GH : LM$ (Cor. 3. 18. Proportion).

PROP. XXVI.

None other but three ſorts of regular plane figures, joined together, can make a ſolid angle; and theſe are, 3, 4, or 5 triangles, 3 ſquares, and 3 pentagons:

And therefore there can only be five regular bodies, the pyramid, cube, octaedron, dodecaedron, and icoſaedron.

Three plane angles at leaſt, are required to make a ſolid angle. One angle of the triangle $= \frac{2}{3}$ of a right angle (2. II), therefore 3 of them put together make two right angles. Alſo 4 of them make $2\frac{2}{3}$ right angles. And 5 make $3\frac{1}{3}$ right angles; all which are leſs than 4 right angles. But 6 of them make 4 right angles, and therefore cannot make a ſolid angle (17. V).

Again, one angle of the ſquare is a right angle, and 3 of them make 3 right angles. But 4 make 4 right angles, and therefore can make no ſolid angle (17. V).

Alſo one angle of the pentagon is $1\frac{1}{5}$ right angle (17. III). And 3 angles make $3\frac{3}{5}$. But 4 of them make $4\frac{4}{5}$, which exceeds 4 right angles.

Laſtly, one angle of the hexagon is $\frac{4}{3}$ of a right angle, therefore 3 angles make 4 right angles; but no ſolid angle. And the angle of a heptagon, octagon, &c. being greater; 3 of them will exceed 4 right angles; and conſequently, there can be no

FIG. 184. more than 3 triangles, 1 ſquare, and 1 pentagon, to conſtitute a ſolid angle.

Hence there can only be 5 regular bodies, to anſwer the 5 combinations of triangles, ſquares, and pentagons. Three faces of the triangle make the *pyramid*; 4 make the *octaedron*; and 5 make the *icoſihedron*; alſo 6 faces of the ſquare make the *cube*; and 12 faces of the pentagon, make the *dodecaedron*.

SCHOLIUM.

185. 186. 187. 188. 189. In order to get a clear idea of the five regular bodies, you may cut out all their faces in paſteboard, as repreſented in the figures, and fold them up, ſo that the creaſes may be in the black lines; and their edges being put cloſe together, you'll have the figure of theſe bodies. Fig. 185 is the pyramid, 186 the cube, 187 the octaedron, 188 the dodecaedron, and 189 the icoſihedron.

PROP. XXVII.

No other but only one ſort, of the five regular bodies, joined at their angles, can compleatly fill a ſolid ſpace; and that is eight cubes.

To demonſtrate this, we muſt obſerve that among other properties, this is abſolutely neceſſary, that the inclination of two adjoining planes in the body, be ſuch; that being taken a certain number of times, they will compleatly make up four right angles. For when the bodies are put together, the faces of every two adjoining bodies muſt coincide; and one edge or ſide of all the bodies muſt coincide with the ſide of the firſt; which will be as an axis, round which theſe bodies are placed; and therefore they muſt compleatly fill up the ſpace quite round, which is four right angles. And the angle of each (that is, the inclination of two adjoining planes), muſt be a certain part of 4 right angles. Therefore what we have to do, is to compute the inclination of their planes, and alſo to enquire what inclination is requiſite in the ſeveral bodies, to have this effect.

1. To begin with pyramids. It is plain, the baſe of the ſolid, being an equilateral triangle, the angle at any point is $\frac{2}{3}$ of a right angle; but the inclination of the planes is greater; for it is contained by two perpendiculars let fall on the common ſection of two planes, which perpendiculars are leſs than the ſides of the triangle (Cor. 4. 21. II); and ſtanding on the ſame baſe, muſt contain a greater angle (Cor. 2. 5. II). 190. To find the inclination of the planes; let CPH, CPA, and CDH be three of the equilateral triangles conſtituting a pyramid. Draw AG, DI $\perp$ to CP, CH. Let the plane CHP be fixt, whilſt the planes CAP, CDH, are raiſed up, (moving about the fixt lines CP, CH,) till the points A and D meet ſomewhere. It is plain a perpendicular dropt from A (elevated on high), upon the plane CPH, will always be ſomewhere in the line AG. And a like perpendicular from D will be ſomewhere in the line DI. Therefore when A and D meet, the perp. will be at the interſection O, in the middle of the triangle; and GO $= \frac{1}{3}$GH (Cor. 31. II) $= \frac{1}{3}$GA. Therefore, if you make the ſeparate right-angled triangle GAO, ſo that the hyp. GA may be treble the baſe GO, the $\angle$AGO is the angle of the pyramid, (that is, of its planes CAP, CHP), which was required. Now if EG be $\perp$ to GK, alſo if GBK be an equilateral triangle, then the baſe GF, will be half the hypothenuſe GB (Cor. 3. 3. II), and $\angle$BGK $= \frac{2}{3}$ a right angle (2. II). Then its plain, 4 times $\angle$AGK will be leſs than 4 right angles, becauſe 4 times EGK make but 4 right angles;

FIG. gles; therefore more than 4 times AGK is required
190. to compleat 4 right angles. Likewiſe, ſince 6 times BGK make 4 right angles, 6 times AGK will be too much; and of conſequence we muſt either have 5 times AGK, to make 4 right angles, or
134. nothing. Then to find whether that will anſwer exactly or not; draw the diagonal EC of the pentagon, and OLD ⊥ to it; then 5 times the angle EOL = 4 right angles. But DL $= \frac{5 - \sqrt{5}}{4}R$ (Cor. 3. 44. IV), and OL $= R - R \times \frac{5 - \sqrt{5}}{4} =$
190. $\frac{\sqrt{5} - 1}{4} \times R$. But GO (fig. 190) $= \frac{1}{3}$ the hypo-
134. thenuſe AG or R, and $\frac{1}{3}$ is greater than $\frac{\sqrt{5} - 1}{4}$, that is, GO is greater than OL, and conſequently the angle AGO is leſſer than EOL, which it ſhould be equal to; therefore 5 times AGO falls ſhort of 4 right angles; whence it is clear, that no combination of regular pyramids can compleatly fill all ſpace.

2. And it is as clear that 4 cubes ſet together will make up 4 right angles, each cube containing one. And therefore 8 cubes, joined at their angular points, will quite fill all ſpace on all ſides.

191. 3. Next for the octaedron. As half the octaedron ABE ſtands on a ſquare baſe BCED, the angles at the baſe, as BCE, are right, and then 4 of theſe would be 4 right ones; but the inclination of the planes ACB, ACE, are greater than right angles (for the ſame reaſon as in the pyramid), being made by a plane ⊥ to their common ſection AC; therefore 4 of theſe angles will be too much, and conſequently 3 or none of theſe angles of inclination muſt be equal to 4 right angles; or, which is the ſame thing, 6 halfs of the ∠ of inclination muſt be = 4 right angles. Now to try this, draw AG ⊥ to

BC, and AO ⊥ to the baſe BE, alſo draw GO. Then FIG. 192. hyp. AG = $\frac{AB}{2}\sqrt{3}$ (Cor. 39. II), and baſe GO = $\frac{1}{2}$BD = $\frac{1}{2}$AB. Therefore AG : GO :: $\frac{AB}{2}\sqrt{3}$: $\frac{AB}{2}$ :: $\sqrt{3} : 1 :: 3 : \sqrt{3} :: 1 : \frac{\sqrt{3}}{3}$ :: AG : $\frac{AG}{3}\sqrt{3}$; then GO = $\frac{AG}{3}\sqrt{3}$. And as ∠AGO = half the angle of inclination, 6 of theſe muſt make up 4 right angles. And therefore ∠AGO muſt be = ∠BDE (fig. 131), if this ſucceed. For 6 of theſe make up 4 right angles. But in this caſe, DF = $\frac{1}{2}$DB, whence if DB (fig. 131) = AG (fig. 191), then GO = $\frac{DB}{3}\sqrt{3}$. But $\frac{1}{3}\sqrt{3}$ is greater than $\frac{1}{2}$ (as is eaſily known by ſquaring them); that is, GO is greater than DF, and conſequently ∠AGO is leſs than BDF. Therefore 6 of theſe, or 3 whole angles of inclination, fall ſhort of 4 right angles. So theſe bodies cannot entirely fill all ſpace.

4. Next comes the dodecaedron. As the angle of inclination of the planes of this body exceeds a right angle; therefore 4 ſuch angles will exceed 4 right angles; therefore only three of theſe bodies can be laid together; in which caſe the angle of inclination muſt be juſt $1\frac{1}{3}$ right angle. For $3 \times 1\frac{1}{3} = 4$. If the ∠ be leſs, the third body will leave a vacuity; if greater, it cannot come in. Let 192. BPG, PCH, DGH, be 3 pentagons joining upon one another. Draw AG, DI ⊥ to PC, HC, continued. Then let the plane PCH, be fixt, whilſt ABP, DEH, are raiſed up, and moved round the lines PC, HC, till the points A, D, meet. It is evident a perpend. dropt from A upon the plane PCH, will always fall on the line AG. And a like perpend. from D, will fall upon DI. And when A and D meet, it will fall on the interſection O.

Let

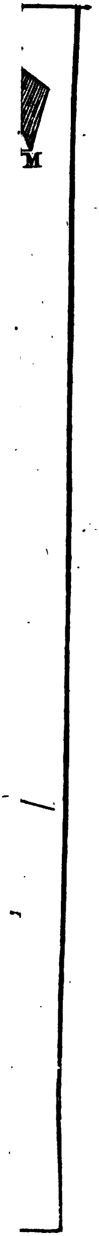
FIG. 192. Let R ſtand for a right angle. Then ſince CE is $\parallel$ to HN (Cor. 2. 43. IV), $\angle$ECH + CHN = 2R (Cor. 2. 4. I) = ECH + PCH, therefore PCE is a right line (1. I). For the ſame reaſon BCH is a right line. Since $\angle$DCH = $\frac{6}{5}$R (17. III), DCE = $\frac{2}{5}$R, DCP = $\frac{8}{5}$R, take away ACP = $\frac{6}{5}$R, then ACD = $\frac{2}{5}$R. In the iſoſceles triangle ACD, COF biſects the $\angle$C and baſe AD (Cor. 3. 3. II), and $\angle$ACF = $\frac{1}{5}$R = DCF, and CDA = $\frac{4}{5}$R; and ſince CDE = $\frac{6}{5}$R, therefore CDA + CDE = 2R, and EDA is a ſtraight line (1. I). In the right-angled triangle ACF, $\angle$ACF or ACO = $\frac{1}{5}$R; and in the right-angled triangle ACG, ſince ACE = ACD + DCE = ACG = $\frac{4}{5}$R, CAG = $\frac{1}{5}$R = ACO, or CAO = ACO, and AO = OC (Cor. 1. 3. II). Therefore OG is leſs than OC or OA (5. II), and OG is leſs than half of AG. Make a right angle triangle ſeparately, as AGO, where the hypothenuſe is AG, and baſe OG, of a due length, and AGR is one of the angles of the dodecaedron. Where the $\angle$AGZ or GAO ought to be $\frac{1}{3}$R, that 3 dodecaedrons laid together may fill up 4 right angles. Now to ſee how this agrees, we find (in fig. 128), that EF = $\frac{1}{2}$DE, or DF = $\frac{1}{2}$DB (Cor. 3. 41. IV), and $\angle$ABF or BAC = $\frac{2}{3}$R = BDF (Cor. 1. 12, IV), and conſequently DBF = $\frac{1}{3}$R (Cor. 2. 2. II). Therefore if you make the baſe GQ = $\frac{1}{2}$ the hypothenuſe GM, then the $\angle$GMQ or MGZ is = $\frac{1}{3}$R. Therefore, ſince GO is leſs than $\frac{1}{2}$GA, the $\angle$AGZ is leſs than $\frac{1}{3}$R, and MGR leſs than $1\frac{1}{3}$R, to which it ſhould have been equal; and conſequently 3 times MGR falls ſhort of 4 right angles: therefore the dodecaedrons cannot fill a ſolid ſpace.

This might be otherwiſe ſolved, by ſuppoſing one of its ſolid angles to ſtand upon an equilateral triangle, whoſe ſide is the diagonal of the pentagon.

5. Laſtly, the icoſaedron has 5 triangles ſtanding upon a pentagonal baſe ABCDE. Draw the diagonal AC of the pentagon, and BQ the diameter of the circumſcribing circle. And let the plane AFC be drawn at right angles to BO, the common ſection of the two faces of the ſolid ABO, CBO. Draw FP, which will be ⊥ to AC. Then we are to find the quantity of the ∠ AFC, the inclination of the planes, or rather, of its half AFP. Call $\frac{1}{2}$BQ, the radius of the circle, R; then AP² (Cor. 2. 44. IV) $= \frac{5+\sqrt{5}}{8}RR$. Alſo $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$ (44. IV), and $AF^2 = \frac{3}{4}AB^2$ (Cor. 39. II) $= \frac{3}{4}RR \times \frac{5-\sqrt{5}}{2}$. Therefore $AF^2 : AP^2 :: \frac{3}{4}RR \times \frac{5-\sqrt{5}}{2} : \frac{5+\sqrt{5}}{8}RR :: 15 - 3\sqrt{5} : 5 + \sqrt{5}$. And $AF^2 : AF^2 - AP^2$ or $FP^2 :: 3 \times \overline{5-\sqrt{5}} : 10 - 4\sqrt{5} :: 3 : \frac{10-4\sqrt{5}}{5-\sqrt{5}} :: 3 : \frac{\overline{10-4\sqrt{5}} \times \overline{5+\sqrt{5}}}{\overline{5-\sqrt{5}} \times \overline{5+\sqrt{5}}} :: 3 : \frac{50+10\sqrt{5}-20\sqrt{5}-20}{25-5=20}$ (Cor. 1. 8. I) $:: 3 : \frac{30-10\sqrt{5}}{20} :: 3 : \frac{3-\sqrt{5}}{2} :: 1 : \frac{3-\sqrt{5}}{6} :: 1 : .12732 :: AF^2 : .12732AF^2$. And by extracting the root, it is $AF : FP :: AF : .3568 \times AF = FP$. Now if three icoſaedrons laid together can fill up the whole ſpace, then three times the angle AFC, or ſix times the ∠ AFP, muſt make four right angles; and in that caſe AFP muſt be $\frac{2}{3}$ of a right angle. But (fig. 128) the ſide DF muſt be half the hypothenuſe DB, when the ∠ between them BDF is $\frac{2}{3}$ of a right angle (Cor. 3. 41. IV): for ∠ BDF = BAC in the equilateral triangle BAC (Cor. 1. 12. IV) = $\frac{2}{3}$ of a right angle (2. II). But here the ſide FP is leſs than half AF or .5 × AF; therefore the ∠ FAP will be leſs, and AFP greater, than it ſhould be; that is,

FIG. 193. is, AFP is more than $\frac{2}{3}$ of a right angle; and 6 times AFP, more than 4 right angles; and therefore 3 icoſihedrons cannot find room.

Thus I have demonſtrated from pure geometrical principles, that no combination of regular bodies of the ſame ſort (except cubes), can adequately fill up all the ſpace round about. The calculations of all theſe caſes are extremely eaſy, by working with the rules of trigonometry; but that was not my buſineſs here.



2.124.

BOOK VII.

Of the ſphere, and its inſcribed and circumſcribed bodies.

DEFINITIONS.

1. A *Sphere or globe*, is a ſolid made by a ſemicircle ABD, moving round about its diameter AD, which remains fixt; and is called the *axis* of the ſphere; and the point A, the *vertex*. FIG. 194.

2. The *center* of the ſphere is the center C of the ſemicircle ABD.

3. The *radius* of the ſphere, is a line drawn from the center to the ſurface of the ſphere.

Cor. *All the radius of a ſphere are equal to one another.*

4. The *diameter* of a ſphere, is a right line drawn from one ſide to the other, through the center.

5. A *ſector* of a ſphere, CFDG, is a part of the ſphere made by the circular ſector FCD, moving round the radius CD. 195.

6. *Segment* of a ſphere, is a part of a ſphere, as FIGD, cut off by a plane FIG. If the plane paſs through the center, that ſegment is a *hemiſphere*.

7. A *zone*, is a part of a ſphere intercepted between two parallel planes.

8. The *middle zone*, is the part between two parallel planes which are equally diſtant from the center.

FIG. 193. 9. A ſolid is ſaid to be *inſcribed* in a ſphere, or a ſphere *circumſcribed* about a ſolid; when all the angles of the ſolid touch the ſurface of the ſphere.

10. A ſphere is ſaid to be *inſcribed* in a ſolid, or a ſolid *circumſcribed* about a ſphere; when the ſphere touches all the planes of the ſolid.

PROP. I.

194. *If a ſphere be cut by a plane* FOG; *the ſection will be a circle.*

Let the two planes CFDG and COD be ⊥ to the cuting plane FOG; then the common ſection CI is ⊥ to the plane FOG (15. V). Draw the line FIG. Then in the triangles CFI, COI, CGI, the ſides CF, CO, CG are equal (Cor. Def. 3), and CI common, and the angles at I right; therefore IF = IO = IG (9. II). Therefore FDG is a circle whoſe center is I (Cor. Def. 3. IV).

PROP. II.

195. *If a ſphere* ABDI *touch a plane* HGL; *a right line* CD, *drawn from the center to the point of contact* D, *is perpendicular to the ſaid plane.*

Let the planes ADB, ADF, cut the touching plane in the lines DH, DG. Then ſince HD, GD, touch the circles BD, FD, (whoſe center is C,) in D, therefore CD is ⊥ to HD, GD (Cor. 2, 10. IV); and therefore CD is ⊥ to the plane HGL (4. V).

PROP. III.

196. *The ſurface of a ſphere is equal to the curve ſurface of its circumſcribing cylinder.*

Let BAP be a hemiſphere, and BHOP a cylinder on the ſame baſe, BTP, and of the ſame altitude.

FIG. 196.

tude. Take IL, an extremely ſmall part of the quadrant BLA, and through L and I, ſuppoſe two planes MLEVQ, and NIFSR to be drawn ⊥ to AC. Through L and I draw the line ILD, and through S and V the line SVG. Then becauſe IL and VS are extremely ſmall; the right lines and arches LI, VS nearly concide. And if the figure DISG be turned about the radius AC; it will generate the fruſtum of a cone; and the ſmall parts of its ſurface ILVS will concide with the portion of the ſpherical ſurface, and be equal thereto (Ax. 8). But the ſurface of the fruſtum ILVS is = IL × half the ſum of the circumferences whoſe diameters are LV and IS (6. VI), that is = IL × circumference of LV or IS, they being nearly equal. Let C = circumference whoſe radius is BC, and c = circumference whoſe radius is LE or IF, then the ſurface ILVS = c × IL, and the cylindric ſurface NMQR = C × MN (4. VI). But the triangles ILK, and LCE are ſimilar; for ∠ILC (Cor. 2. 10. IV) = KLE = a right angle; take away KLC, then ∠ILK = CLE; alſo ∠IKL = LEC = a right angle. Therefore LC or BC : LE :: LI : LK (13. II). But C : c :: BC or ME : LE (Cor. 9. IV) :: LI : LK or MN. Whence C × MN = c × IL (12. Proportion); that is, the cylindric ſurface NMQR = ſpherical ſurface ILVS. Therefore if more parallel planes, as MLVQ, be drawn, exceeding near to one another, the ſmall parts of the cylindric ſurface will be equal to the correſpondent parts of the ſpherical ſurface, and therefore the ſum of all the parts of the cylindric ſurface, equal to the ſum of all the parts of the ſpherical ſuface (Ax. 2); that is, the ſurface of the hemiſphere is equal to the ſurface of the cylinder BO, and the ſurface of the whole ſphere = ſurface of its circumſcribing cylinder.

Cor.

FIG. 196. Cor. 1. *If the ſphere and its circumſcribing cylinder be cut by two planes parallel to the baſe, the intercepted parts of the ſurfaces of the ſphere and cylinder, will be equal.*

For ſurface MR = ſurface LS, and all the MR = all the LS.

Cor. 2. *The ſurface of the hemiſphere* BAP, *is double the baſe* BTP.

For the ſurface of the cylinder = C × AC (4. VI); and the area of the baſe = $\frac{C \times BC}{2}$, or $\frac{C \times AC}{2}$ (34. IV).

Cor. 3. *The ſurface of the whole ſphere is equal to four great circles of the ſame ſphere; or to the rectangle of the circumference and diameter.*

Cor. 4. *The areas of ſpherical ſurfaces cut off by parallel planes, are as the ſegments of the diameter, perpendicular thereto.*

For theſe areas are equal to the correſponding cylindrical ſurfaces, which are as the hights (Cor. 3. 4. VI).

Cor. 5. *The ſurface of any ſegment of the ſame ſphere, is as the hight of the ſegment.*

Cor. 6. *The ſurface of the ſphere is* $\frac{2}{3}$ *the whole ſurface of the circumſcribing cylinder.*

For the two baſes of the cylinder is half its curve ſurface (Cor. 3).

PROP. IV.

197. *The ſurface of the ſegment* BAD, *of a ſphere; is*
198. *equal to the area of a circle, whoſe radius is the cord* AB, *drawn from the vertex to the baſe.*

Let

FIG. 200. Let C = circumference of the radius AB, and ABED = circumference of the ſphere. Then ſince the circumferences are as the radii (Cor. 9. IV), let $\frac{ABED}{AC} = \frac{C}{AB} = n$, or $ABED = n \times AC$, and $C = n \times AB$. Then the ſurface BAD $= AF \times ABED$ (Cor. 1. III) $= AF \times n \times AC = \frac{n \times AF \times AE}{2} = \frac{n \times AB^2}{2}$ (Cor. 17. IV) $= \frac{AB \times C}{2}$ = area of a circle whoſe radius is AB (34. IV).

Cor. *The ſurface of the whole ſphere, is equal to the area of a circle, whoſe radius is the diameter* AE.

PROP. V.

201. *The ſurface of a ſphere is double the curve ſurface of the inſcribed ſquare (or equilateral) cylinder* EB.

Draw the diameter ECB, then ED = DB. And ſince $EB^2 = ED^2 + DB^2$ (21. II) $= 2ED^2$; therefore circle EB = 2 circles ED (22. III). But ſurface of the ſphere = 4 circles EB (Cor. 3. III) = 8 circles ED. And $\frac{1}{4}$ED : AE or ED :: circle ED : curve ſurface AD (Cor. 2, 4. VI) = 4 circles ED. But 8 circles ED = twice 4 circles ED, or the ſurface of the ſphere = twice the curve ſurface of the cylinder.

Cor. 1. *The whole ſurface of the inſcribed cylinder is $\frac{3}{4}$ the ſurface of the ſphere.*

For the two baſes AB, ED = 2 circles ED, and the whole ſurface AD = 6 circles ED.

Cor. 2. *The curve ſurface of a cylinder, circumſcribing the ſphere, is double the curve ſurface of the inſcribed equilateral one. And the whole ſurface, is double to the whole ſurface.*

FIG. 201. For the ſurface of the ſphere = ſurface of the circumſcribing cylinder (3). And the ſurface of the ſphere = twice the ſurface of the inſcribed one (5).

Again, the ſurface of the ſphere = $\frac{2}{3}$ the whole ſurface of the circumſcribing cylinder (Cor. 6. 3). And the ſurface of the ſphere is = $\frac{4}{3}$ the whole ſurface of the inſcribed cylinder (Cor. 1).

PROP. VI.

202. *The ſurface of any ſegment of a ſphere* ABDC : *is to the curve ſurface of its inſcribed cone* ABC : : *as the ſide of the cone* AB : *to the radius of the baſe* AO.

For if $n \times AB$ = circumference of the radius AB, and $n \times AO$ = circumference of the radius AO (Cor. 9. IV), then the circle AB $= \frac{AB^2 \times n}{2}$ (34. IV), and conic ſurface ABC $= \frac{AB \times n \times AO}{2}$ (Cor. 1. 5. VI). And the ſurface of the ſegment ABDC = circle AB (4); therefore ſurface of the ſegment ABDC : conic ſurface ABC : : $\frac{AB^2 \times n}{2}$: $\frac{AB \times n \times AO}{2}$: : AB : AO (5. Proportion).

Cor. 1. *The ſurface of a hemiſphere, is to the curve ſurface of its inſcribed cone; as the diagonal of a ſquare, to the ſide.*

For then AO, BO become radii of the ſphere, and AB the diagonal.

Cor. 2. *If* ABC *be an equilateral cone, then the ſurface of the ſegment* ABDC *is twice the curve ſurface of the cone* ABC.

For then AB = AC = 2AO.

PROP. VII. FIG.

Let the cone DAE *be right-angled at* A. *Then the ſurface of the hemiſphere* BGE, *is to the curve ſurface of the right-angled circumſcribing cone* DAE; *as the ſide of a ſquare* AD, *is to the diagonal* DE. 203.

Draw AC from the vertex of the cone A, to the center C; and CF ∥ to AE, or ⊥ to AD. Then AF = FD = FC = BC, and $CD^2 = CF^2 + FD^2 = 2BC^2$ (21. II). And the circle whoſe radius is CD = twice the circle whoſe radius is CB (Cor. 2. 35. IV) = ſurface of the hemiſphere BGE (Cor. 2. 3). Therefore the ſurface of the hemiſphere, or the circle whoſe radius is CD : ſurface of the cone DAE :: CD : AD (Cor. 3. 5. VI) :: AD : DE (20. II).

Cor. *The ſurface of a right-angled cone circumſcribing a hemiſphere, is double the ſurface of one inſcribed; taking either the curve ſurfaces, or the whole ſurfaces.*

For $\sqrt{2}$ × ſurface of the inſcribed cone = ſurface of the hemiſphere (Cor. 1. 6) = $\frac{1}{\sqrt{2}}$ × ſurface of the circumſcribing cone (7). Therefore the latter is = twice the former. And the baſe of the latter is likewiſe = twice the baſe of the former (by the demonſtration of this Prop.), therefore the whole is double to the whole.

PROP. VIII.

The ſurface of the ſphere, is to the curve ſurface of an equilateral inſcribed cone BAD; *as* 8, *to* 3. 204.

For ſince EF = $\frac{1}{4}$AF (Cor. 3. 41. IV), therefore ſurface BFD = $\frac{1}{4}$, and ſurface BAGD = $\frac{3}{4}$, the ſurface of the ſphere (Cor. 4. 3), = 2 curve

FIG. 204. ſurfaces of the cone BAD (Cor. 2. 6); or the ſurface of the cone = $\frac{3}{8}$ the ſurface of the ſphere.

Cor. *The whole ſurface of an equilateral cone* BAD, *inſcribed in a ſphere, is* $\frac{9}{16}$ *of the ſphere's ſurface.*

For $3BC^2 = BD^2$ (41. IV) $= 4BE^2$, and $BE^2 = \frac{3}{4}BC^2$, whence circle BD = $\frac{3}{4}$ circle BDG (35. IV) = $\frac{3}{16}$ the ſurface of the ſphere (Cor. 3. 3); add this to the curve ſurface of the cone; then the whole ſurface of the cone = $\frac{3}{8} + \frac{3}{16}$ the ſphere's ſurface = $\frac{9}{16}$ the ſurface of the ſphere.

PROP. IX.

205. *The curve ſurface of an equilateral cone* ABD, *is to the ſurface of its inſcribed ſphere; as 3 to 2.*

Draw AE, CF ⊥ to BD, BA; then by ſimilar triangles AEB, AFC; $AE^2 : EB^2 :: AF^2 : FC^2$. But $AE^2 = \frac{3}{4}AB^2$ (39. II) $= 3AF^2$. Therefore $3AF^2$ (AE^2) : $AF^2 :: EB^2 : FC^2$ (4. Proportion) :: circle BD : circle FEG. But BE :: BA or 2BE :: circle BD : curve ſurface of the cone BAD (Cor. 3. 5. VI) = 2 circles BD; and circle FEG = $\frac{1}{4}$ ſurface of the ſphere (Cor. 3. 3). Whence $3 : 1 :: 3AF^2 : AF^2 :: \frac{1}{2}$ ſurface of the cone : $\frac{1}{4}$ ſurface of the ſphere. Therefore the ſurface of the ſphere = $\frac{2}{3}$ the curve ſurface of the cone.

Cor. 1. *The ſurface of the ſphere is* $\frac{4}{9}$ *the whole ſurface of the circumſcribing equilateral cone.*

For the baſe BD = $\frac{1}{2}$ curve ſurface of the cone = $\frac{3}{4}$ ſurface of the ſphere. Add this to the curve ſurface, which is = $\frac{3}{2}$ ſurface of the ſphere; then the whole ſurface of the cone = $\frac{3}{2} + \frac{3}{4}$ the ſurface of the ſphere = $\frac{9}{4}$ the ſurface of the ſphere, or $\frac{4}{9}$ the whole ſurface of the cone = the ſurface of the ſphere.

Cor. 2. *The curve ſurface of an equilateral cone inſcribed in a ſphere is = $\frac{1}{4}$ the curve ſurface of the circumſcribing equilateral one. And the whole ſurface of one = $\frac{1}{4}$ the whole ſurface of the other.* FIG. 205.

For $\frac{8}{3}$ the ſurface of the inſcribed cone = ſurface of the ſphere (8) = $\frac{2}{3}$ ſurface of the circumſcribed cone (9). Therefore the ſurface of the inſcribed = $\frac{1}{4}$ the ſurface of the circumſcribed one.

Alſo $\frac{4}{9}$ the whole ſurface of the circumſcribing one = ſurface of the ſphere (Cor. 1. 9) = $\frac{16}{9}$ the whole ſurface of the inſcribed cone (Cor. 8). Therefore the ſurface of the inſcribed cone = $\frac{1}{4}$ the ſurface of the circumſcribed cone.

Cor. 3. *The ſurfaces of a cylinder and equilateral cone, both circumſcribed about a ſphere, are as 2 to 3; both their curve ſurfaces and whole ſurfaces.*

For $\frac{2}{3}$ the curve ſurface of the cone = ſurface of the ſphere (9) = ſurface of the cylinder (3). Surface of the cylinder : ſurface of the cone : : 2 : 3.

Alſo $\frac{4}{9}$ the whole ſurface of the cone = ſurface of the ſphere (Cor. 1. 9) = $\frac{2}{3}$ the whole ſurface of the cylinder (Cor. 6. 3). Therefore, whole ſurface of the cylinder : whole ſurface of the cone : : $\frac{4}{9}$: $\frac{2}{3}$ or $\frac{6}{9}$: : 2 : 3.

SCHOLIUM.

From the foregoing propoſitions are deduced, the proportion of the ſphere's ſurface, to the ſurfaces of the inſcribed and circumſcribed equilateral cylinder and cone, as follows:

FIG. 205.

Surface of the ſphere ——— ———	16
Inſcribed cylinder's curve ſurface —	8
——— ——— whole ſurface —	12
Circumſcribed cylinder's curve ſurface	16
——— ——— whole ſurface —	24
Inſcribed cone's curve ſurface ———	6
——— ——— whole ſurface —	9
Circumſcribing cone's curve ſurface —	24
——— ——— whole ſurface —	36.

PROP. X.

206. *A ſphere is equal to a cone whoſe hight is the radius* AC, *and baſe the ſurface of the ſphere* AEF.

Take three points in the ſurface of the ſphere, as A, B, D, extremely near together, forming the ſmall triangle ABD, on the ſurface of the ſphere. Let a plane paſs through theſe three points A, B, D; the ſmall portion of which ABD will coincide with a portion of the ſpherical ſurface ABD, extremely near. And the radius CA will be ⊥ thereto (2). Therefore the portion of the ſphere CABD is nothing but the pyramid whoſe baſe is ABD, a ſmall part of the ſphere's ſurface, and hight the radius CA. In like manner the whole ſphere may be divided into ſmall pyramids, ſuch as CABD, whoſe baſe is a ſmall portion of the ſpherical ſurface; and common altitude, the radius CA. Therefore the ſum of all theſe pyramids CABD, make up the ſphere; and the ſum of all the baſes ABD, make up the ſpherical ſurface. That is, the ſphere is equal to the ſum of all theſe pyramids, whoſe baſes are all the parts of the ſurface, of the ſphere, and common altitude the radius CA; and that is equal to one pyramid or cone, whoſe baſe is the ſurface of the ſphere, and hight the radius (Ax. 2).

Cor. 1. *A ſphere is equal to a cone, whoſe hight is the radius, and baſe equal to four great circles of the ſphere.* FIG. 206.

For the ſurface of the ſphere is equal to four great circles (Cor. 3. 3).

Cor. 2. *A ſphere is equal to a cone whoſe hight is twice the diameter, and baſe, a great circle of the ſphere.*

By Cor. 4. 20. VI.

Cor. 3. A *hemiſphere is double its inſcribed cone.*

For a hemiſphere = a cone whoſe baſe is a great circle, and hight equal to the diameter (Cor. 2); and that is double to a cone of the ſame baſe, and half the hight (Cor. 1. 20. VI).

PROP. XI.

Any ſphere BANR, *is* $\frac{2}{3}$ *its circumſcribing cylinder,* DM. 207.

Let AC be the axis of the hemiſphere BAN. From the center C, draw the diagonal CD; and draw PL $\perp$ to AC, and OH parallel to it, and exceeding near it. Then if the figure ADBC revolve round the axis AC; then ADBC will generate the cylinder BDGN; the quadrant BVA, the hemiſphere BAN; and ADC, the cone ADCG. Then $VC^2 = VL^2 + LC^2$ (21. II); that is, $PL^2 = VL^2 + KL^2$ (for DA = AC, and KL = LC (13. II). Therefore the circle deſcribed by LP = the two circles deſcribed by LV and LK (Cor. 2. 35. IV). Take away the circle deſcribed by LV, from both, and there remains the annulus or ring deſcribed by VP = circle deſcribed by LK. For the ſame reaſon the annulus deſcribed by OI = circle deſcribed by FH. Therefore the ſmall priſmatic ſolid contained between PN and OI, quite round the figure = cone fruſtum contained between KL and FH, round the

FIG. 207. figure (12. VI). In like manner every part of the figure BDAVB = correſpondent part of DACG. Therefore the total ſum of the firſt = total ſum of the laſt, that is, the ſolid BDAGNAVB = cone DCG (Ax. 2) = $\frac{1}{3}$ the cylinder DBNG (20. VI). Therefore the remaining part, or the hemiſphere BAN = the remaining $\frac{2}{3}$ of the cylinder BDGN. Whence the double thereof, or the whole ſphere ABRN = $\frac{2}{3}$ of the whole cylinder EG.

Otherwiſe.

The cone whoſe baſe is BN, and hight CA, or the cone DCG = half the hemiſphere BAN (Cor. 3. 10). And the ſame cone DCG = $\frac{1}{3}$ the cylinder BDGN, (20. VI). Therefore $\frac{1}{2}$ hemiſphere = $\frac{1}{3}$ cylinder, and the hemiſphere = $\frac{2}{3}$ cylinder BG. Whence the whole ſphere = $\frac{2}{3}$ the cylinder EG.

Cor. 1. *The concave ſolid* BFADBER *&c.* = $\frac{1}{2}$ *the ſphere* BANR.

208. Cor. 2. *A right cone, ſphere, and cylinder, all of the ſame diameter and hight, are as* 1, 2, 3 *reſpectively; or* ABD : AHGI : EBDF : : 1 : 2 : 3.

PROP. XII.

206. *The ſector of a ſphere* CGAH, *is equal to a cone whoſe hight is the radius; and baſe, the ſurface of the ſector* GAH.

This is demonſtrated as Prop. X. For if the ſector be divided into a multitude of extremely ſmall ſectors CABD, the baſe of each will be a ſmall portion of the ſpherical ſurface ABD. And as all the pyramids make up the ſector, and are the elements thereof; ſo all the baſes are the elements of the ſurface GAH, and make it up. And as the hights of all the pyramids is the ſame, they are all equal to one pyramid of the ſame hight, and baſe the ſum of all the baſes (Cor. 1. 18. VI). That is, the ſector CGAH = a pyramid or cone whoſe hight is the radius, and baſe the ſurface GAH. FIG. 206.

Cor. 1. *The ſector of a ſphere,* CGAH = *a cone, whoſe hight is the radius* AC; *and baſe a circle whoſe radius is* AG. *And the ſector* CGBH = *a cone whoſe radius is* CB, *and baſe a circle whoſe radius is* BG.

For the ſurface GAH = a circle whoſe radius is AG (4); and the ſurface GBH = a circle whoſe radius is BG (ibid.).

Cor. 2. *Sectors of ſpheres, are to one another, in the complicate ratio of their ſurfaces and radii.*

For the cones, equal thereto, are as the baſes and hights (Cor. 3. 20. VI).

PROP. XIII.

If it be made, as BD : BA :: *radius* CA : CF; *then the cone* GFH *is equal to the ſegment of the ſphere,* GAH. 210.

Draw CG, BG and FCB; then CA : CF :: BD : BA (hyp.) :: $BD^2 : BG^2$ (Cor. 1. 20. II) :: $GD^2 : GA^2$ (20. II) :: circle GD (or circle whoſe radius is GD) : circle GA (35. IV). Therefore the cone whoſe hight is CF, and baſe the circle GD = cone whoſe hight is CA, and baſe the circle GA (Cor. 4. 20. VI) = ſector CGAH (Cor. 1. XII). Subtract, or add the cone GCH, on the ſame baſe GH, and then the cone GFH = ſegment GAH.

Cor. 1. *If* BD : DA :: *radius* CA : AF. *Then the cone* GFH = *ſegment* GAH.

For ſince BD : BA :: CA : CF, therefore BD : BA — BD :: CA : CF — CA (13. Proportion); that is, BD : DA :: CA : AF.

Cor. 2. *The ſegment* GAH, *is to the inſcribed cone* GAH; *as* FD *to* AD.

Cor.

FIG. 210. Cor. 3. *The ſegment* GAH : *ſegment* GBH : : $\overline{GC+DB}\times AD^2 : \overline{GC+AD}\times DB^2$.

For the hight of the cone, equal to the ſegment GAH, that is, $DF = \frac{GC}{DB}\times DA + DA$ (Cor. 1) $= \frac{GC+DB}{DB}\times DA$. And in like manner, the hight of the cone equal to the ſegment GBH, is $\frac{GC+DA}{DA}\times DB$. And theſe cones are as the altitudes (Cor. 1. 20. VI); that is, as $\frac{GC+DB}{DB}\times DA$, and $\frac{GC+DA}{DA}\times DB$, or as $\overline{GC+DB}\times DA^2 : \overline{GC+DA}\times DB^2$.

PROP. XIV.

210. *The ſegment of a ſphere* GAH, *is equal to a cone, whoſe hight is* AD, *the hight of the ſegment; and baſe,* $\frac{1}{2}$ *the baſe of the ſegment* GH, *together with* $\frac{1}{2}$ *a circle whoſe radius is the hight of the ſegment* AD.

Let $\odot AG$ denote the circle whoſe radius is AG, and ſo of the reſt. Then ſegment GAH = ſector CGAH $\mp$ cone GCH (fig. 1, 2) $= \frac{1}{3}AC\times\odot AG \mp \frac{1}{3}CD\times\odot GD$ (Cor. 1. 12); and 3 ſegments GAH $= AC\times\odot AG + \overline{AD\mp AC}\times\odot GD = AC\times\overline{\odot AG - \odot GD} + AD\times\odot GD = AC\times\odot AD + AD\times\odot GD$ (Cor. 2. 35. IV).

But $AD : AB :: AD^2 : AG^2$ (Cor. 1. 20. II) $:: AD^2 : AD^2 + DG^2 :: \odot AD : \odot AD + \odot DG$ (Cor. 2. 35. IV), therefore $AD\times\overline{\odot AD + \odot DG} = AB\times\odot AD = 2AC\times\odot AD$, and $AD\times\overline{3\odot DG + \odot AD} = 2AC\times\odot AD + 2AD\times\odot GD$. And $AD\times\overline{1\frac{1}{2}\odot GD + \frac{1}{2}\odot AD} = AC\times\odot AD + AD\times\odot GD =$ 3 ſegments GAH.

Corol.

Corollary. *The ſegment* GAH $= \frac{1}{6}AD \times \overline{3\Theta GD + \Theta AD}$. FIG. 210.

PROP. XV.

211. *The fruſtum or middle zone of a ſphere* ZGHF, *is equal to a cone whoſe hight is the hight of the zone* CD; *and baſe, two great circles* ZF, *together with the leſſer baſe* GH.

For the zone ZH = hemiſphere ZAF — the ſector CGAH + the cone GCH $= AC \times \frac{2}{3}\Theta ZC$ (11) $- AC \times \frac{1}{3}\Theta AG$ (Cor. 1. 12) $+ CD \times \frac{1}{3}\Theta GD$ (20. VI) $= AD \times \frac{2}{3}\Theta ZC + DC \times \frac{2}{3}\Theta ZC - AC \times \frac{1}{3}\Theta AG + CD \times \frac{1}{3}\Theta GD$. But $AD : AC :: AG^2 : AZ^2$ (18. IV) $:: AG^2 : 2AC^2$ (21. II) $:: \Theta AG : 2\Theta ZC$ (35. IV). Therefore $AD \times 2\Theta ZC = AC \times \Theta AG$. And $AD \times \frac{2}{3}\Theta ZC = AC \times \frac{1}{3}\Theta AG$. Therefore the zone $ZH = DC \times \frac{2}{3}\Theta ZC + DC \times \frac{1}{3}\Theta GD = \frac{1}{3}DC \times \overline{2\Theta ZC + \Theta GD}$.

Cor. *The zone* ZH *is equal to* $\frac{1}{3}$DC × *twice the circle* ZF + *the circle* GH.

PROP. XVI.

An orb or hollow ſphere is equal to the fruſtum of a cone, whoſe greater baſe is the ſurface of the greater ſphere; and leſſer baſe, the ſurface of the leſſer: and hight, the difference of the radii.

For the orb is equal to the difference of the two ſpheres; that is, to the difference of two cones whoſe hights are the radii of the ſpheres, and baſes the ſurfaces (10).

PROP.

FIG.

PROP. XVII.

212. 213. *The ſurfaces of ſpheres* GH, IK, *are as the ſquares of the diameters*, AB, DF.

For the ſurface of the ſphere GH = 4 circles AGBH, and the ſurface of the ſphere IK = 4 circles IDKF (Cor. 3. III). But 4 circles AGBH : 4 circles DIFK : : circle AGBH : circle DIFK (Cor. 1. 5. Proportion) : : $AB^2 : DF^2$ (35. IV).

PROP. XVIII.

Spheres GH, IK, *are to one another, as the cubes of their diameters*, AB, DF.

For the ſphere GH = $\frac{2}{3}$ the cylinder, whoſe baſe is AGBH, and hight AB. And the ſphere IK = $\frac{2}{3}$ the cylinder, whoſe baſe is DIFK, and hight DF (11). Therefore ſphere GH : ſphere IK : : $\frac{2}{3}$AGBH × AB : $\frac{2}{3}$DIFK × DF (Cor. 3. 19. VI) : : AGBH × AB : DIFK × DF (5. Proportion) : : $AB^2 \times AB : DF^2 \times DF$ (35. IV. and 7. Proportion) : : $AB^3 : DF^3$.

PROP. XIX.

214. 215. *Similar ſolids inſcribed in ſpheres* GH, IK, *are as the cubes of the diameters of the ſpheres* AB : DF.

From any two equal and correſpondent angles A, D, draw the diameters AB, DF. Then ſince the ſolids are inſcribed after a ſimilar manner in reſpect of the diameters AB, DF. It will be AG : DI : : AB : DF (19. III). But ſolid AE : ſolid DL : : $AG^3 : DI^3$ (24. VI) : : $AB^3 : DF^3$ (Cor. 3. 18. Proportion).

Cor. 1. *Similar ſolids inſcribed in ſpheres, are as the ſpheres.*

For ſpheres are alſo as the cubes of their diameters (18).

Cor. 2. *The ſurfaces of ſimilar ſolids inſcribed in ſpheres, are as the ſquares of the diameters of the ſpheres.* FIG. 214. 215.

For ſurface of AE : ſurface of DL : : $AG^2 : DI^2$ (7. VI) : : $AB^2 : DF^2$.

Cor. 3. *The ſurfaces of ſimilar ſolids inſcribed in ſpheres, are as the ſurfaces of the ſpheres.*

For they are both as the ſquares of the diameters (17).

PROP. XX.

A ſphere, is to any circumſcribing ſolid BF, *(all whoſe planes touch the ſphere)*; *as the ſurface of the ſphere, to the ſurface of the ſolid.* 216.

Since all the planes touch the ſphere, the radius drawn to all the points of contact, will be ⊥ to each plane (2). Therefore if planes be drawn through the center C of the ſphere, and through all the ſides of the body; then the body will be divided into pyramids, BCAE, BCAD, *&c.* whoſe baſes are the planes BAE, BAD, *&c.*; and their common altitude CP, the radius of the ſphere. And the ſum of all theſe pyramids, or the whole ſolid, is equal to a pyramid or cone, whoſe baſe is the ſum of all the plane figures, and hight the radius CP (Cor. 1. 18. and Cor. 2. 20. VI). But the ſphere is alſo equal to a cone or pyramid whoſe baſe is the ſurface of the ſphere, and hight the ſame radius CP (10). And this laſt cone : former cone : : baſe of the latter : baſe of the former (Cor. 2. 20. VI,); that is, the ſphere : circumſcribing ſolid : : ſurface of the ſphere : ſurface of the ſolid.

Cor. 1. *All circumſcribing cylinders, cones,* &c. *are to the ſphere, as their ſurfaces are.*

For

FIG. 216. For any cylinder, or cone, may be conceived to be made up of an infinite number of ſmall planes, all of which touch the ſphere.

Cor. 2. *All bodies circumſcribing the ſame ſphere, are to one another as their ſurfaces.*

Cor. 3. *The ſphere is the greateſt or moſt capacious of all bodies of equal ſurface.*

For if the planes be ſuppoſed to touch the ſphere, their areas will be greater than the ſurface of the ſphere, which is contrary to the hypotheſis; therefore the planes muſt fall within the ſphere; and then the perpendicular upon them will be ſhorter than the radius, and therefore the body will be leſs than the ſphere, as having the ſame baſe, and a leſs hight.

PROP. XXI.

210. *Any ſegment of a ſphere* GAH, *is to its inſcribed cone; as* BC + BD, *to* BD.

For if $AF = \frac{AD}{DB} \times AC$, then the ſegment GAH = cone GFH (Cor. 1. 13). Therefore $FD = \frac{AD}{DB} \times AC + AD$. And this cone GFH : cone GAH :: DF : DA (Cor. 1. 20. VI) :: $\frac{AC}{BD} \times AD + AD : AD :: \frac{AC + BD}{BD} \times AD : AD ::$ AC + BD : BD (5. Proportion).

Cor. 1. *A hemiſphere is double the inſcribed cone.*

For then BD = AC or BC.

Cor. 2. *The ſegment containing an equilateral cone, is equal to 3 times the cone.*

For then BD = $\frac{1}{2}$BC (Cor. 3. 41. IV).

PROP.

PROP. XXII.

FIG. 203.

If the cone DAE *circumſcribing a hemiſphere be right-angled at* A; *that cone* DAE *is to the inſcribed hemiſphere; as* $\sqrt{2}$ *to* 1.

For let $\odot$ ſtand for circle, then ſuppoſing the ſame conſtruction as in Prop. VII, then we have $CD^2 = 2BC^2$, and $CD = BC\sqrt{2} = DF\sqrt{2}$, and $CD : DF :: \sqrt{2} : 1$ (Cor. 1. 12. Proportion); alſo $\odot CD = 2\odot CB$, and $AC = CD$. The cone DAE $= \odot CD \times \frac{1}{3}AC$ (20. VI) $= 2\odot CB \times \frac{1}{3}CD$. Alſo the hemiſphere $= \frac{2}{3}\odot CB \times GC$ (11) $= 2\odot CB \times \frac{BC}{3}$. Therefore the cone : hemiſphere :: $2\odot CB \times \frac{1}{3}CD : 2\odot CB \times \frac{1}{3}BC :: CD : CB$ or $DF :: \sqrt{2} : 1$.

Cor. *A right-angled cone, circumſcribing a hemiſphere, is to the inſcribed cone; as* $2\sqrt{2}$ *to* 1.

For the circumſcribed cone : hemiſphere :: $\sqrt{2} : 1 :: 2\sqrt{2} : 2$ (22).

And hemiſphere : inſcribed cone :: 2 : 1 (Cor. 1. 21).

Therefore circumſ. cone : inſ. cone :: $2\sqrt{2} : 1$ (15. Proportion).

PROP. XXIII.

201.

A ſphere is to its inſcribed equilateral cylinder AD, *as* $4\sqrt{2}$ *to* 3.

Draw the diameter BE, then $BE^2 = DE^2 + DB^2$ (21. II) $= 2DE^2$, and circle AEDB = 2 circles BD (35. IV); alſo $BE = DE\sqrt{2} = BD\sqrt{2}$. Now The ſphere $= \frac{2}{3}AEDB \times BE$ (11) $= \frac{2}{3}AEDB \times BD\sqrt{2}$, the cylind. = circle $ED \times BD = \frac{1}{2}AEDB \times BD$. Then ſphere : cylinder :: $\frac{2}{3}AEDB \times BD\sqrt{2} : \frac{1}{2}AEDB \times BD :: \frac{2}{3}\sqrt{2} : \frac{1}{2} :: 4\sqrt{2} : 3$.

Cor.

FIG. 201. Cor. *The circumſcribed equilateral cylinder, is to the inſcribed equilateral cylinder; as* $2\sqrt{2}$ *to* 1.

For $\frac{2}{3}$ the circumſcr. cylinder = ſphere (11) = $\frac{4\sqrt{2}}{3}$ × the inſcr. cylinder. Therefore the circumſ. cylinder = $2\sqrt{2}$ × inſcr. cylinder.

PROP. XXIV.

204. *The ſphere is to the inſcribed equilateral cone* BAD, *as* 32 *to* 9.

Let ⊙BE denote the circle whoſe radius is BE, *&c.* then BD^2 or $4BE^2 = 3BC^2$ (41. IV), and $BE^2 = \frac{3}{4}BC^2$, and ⊙BE = $\frac{3}{4}$⊙BC (Cor. 1. 35. IV). Alſo AE = $\frac{3}{2}$AC (Cor. 3. 41. IV). Then the ſphere = $\frac{2}{3}$⊙BC × 2AC (11). And cone = ⊙BE × $\frac{1}{3}$AE (20. VI) = $\frac{3}{4}$⊙BC × $\frac{1}{3}$ × $\frac{3}{2}$AC. Therefore, ſphere : cone :: $\frac{2}{3}$⊙BC × 2AC : $\frac{3}{4}$⊙BC × $\frac{1}{2}$AC :: $\frac{4}{3}$: $\frac{3}{8}$ :: 32 : 9.

PROP. XXV.

205. *A ſphere is to its circumſcribed equilateral cone* ABD, *as* 4 *to* 9.

The conſtruction of Prop. IX. remaining; let ⊙FC denote the circle whoſe radius is FC, *&c.* Then $EB^2 = 3FC^2$, and ⊙BE = 3⊙FC (35. IV), and CF or CE = $\frac{1}{2}$CA (Cor. 31. II), and AE = 3CF.

The ſphere = $\frac{2}{3}$⊙CF × 2CF (11).

The cone = ⊙BE × $\frac{1}{3}$ AE (20. VI) = 3⊙FC × FC.

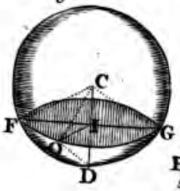
Therefore ſphere : cone :: $\frac{2}{3}$⊙CF × 2CF : 3⊙CF × CF :: $\frac{4}{3}$: 3 :: 4 : 9.

Cor. 1. *The circumſcribed equilateral cone is eight times the inſcribed equilateral cone.*

For the circumſcr. cone : ſphere :: 9 : 4.
And ſphere : inſcr. cone :: 32 : 9 (24).
Therefore circumſcr. cone : inſcr. cone :: 32 : 4 (15. Proportion) :: 8 : 1.

Cor.

196.



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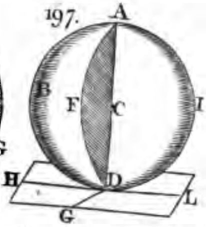

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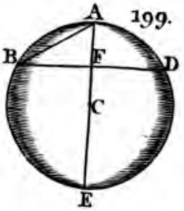

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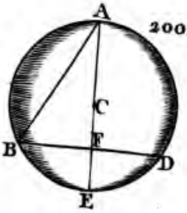

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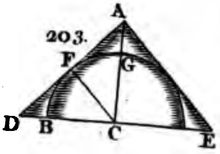
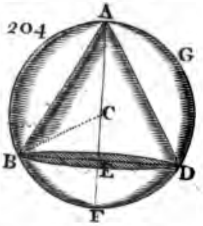
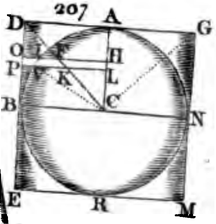


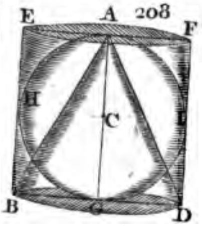
204

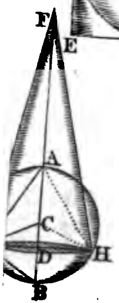


207

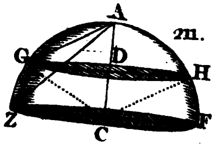


208





211.



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UNIV.

Cor. 2. *The circumſcribed cylinder is $\frac{2}{3}$ the circumſcribed equilateral cone.* FIG. 205.

For $\frac{2}{3}$ the cylinder = ſphere (11) = $\frac{4}{9}$ the cone; and the cylinder = $\frac{2}{3}$ the cone.

Cor. 3. *The ſphere* EF *is to the circumſcribing right cylinder* BC, *and this cylinder to the circumſcribing equilateral cone* ADG, *as* 2 *to* 3; *both in reſpect of their whole ſurfaces and ſolidities.* 217.

This appears from Cor. 6. 3. and Cor. 3. 9. and Prop. 11. and Cor. 2. 25.

Cor. 4. *The circumſcribing right cylinder, and equilateral cone, are to one another as* 2 *to* 3; *both in regard to their curve ſurfaces, their whole ſurfaces, ſolidities, baſes, and hights.*

As to the ſurfaces it appears by Cor. 3. 9; and the ſolidities, by Cor. 3. of this. As to the baſes, ſince $\odot BE = 3\odot FC$ (fig. 205), or $\odot FC = \frac{1}{3}\odot BE$, therefore $2\odot FC = \frac{2}{3}\odot BE$, or the two baſes of the cylinder = $\frac{2}{3}$ the baſe of the cone.

And for the hight, $AE = 3CF$, or $2CF = \frac{2}{3}AE$; that is, the hight of the cylinder = $\frac{2}{3}$ the hight of the cone.

SCHOLIUM.

From the foregoing propoſitions, is eaſily deduced the proportion which the ſphere has to the inſcribed and circumſcribed equilateral cylinders and cones, as follows:

Solidity of the ſphere	32
——————— inſcribed cone	9
——————— inſcribed cylinder	$12\sqrt{2}$
——————— circumſcribed cylinder	48
——————— circumſcribed cone	72.

PROP. XXVI.

FIG. 218. *The ſquare of the ſide of a regular pyramid inſcribed in a ſphere, is $\frac{2}{3}$ the ſquare of the diameter:* $AE^2 = \frac{2}{3}EF^2$.

For drawing ECF $\perp$ to the baſe ABD, $3AC^2 = AB^2$ (41. IV) $= AE^2 = AC^2 + CE^2$ (21. II); and $2AC^2 = CE^2$, or $\frac{1}{2}CE^2 = AC^2 = EC \times CF$ (17. IV), therefore $CF = \frac{1}{2}CE$, and $EF = \frac{3}{2}CE$, or $CE = \frac{2}{3}EF$, and $AC^2 = \frac{1}{2}CE^2 = \frac{2}{9}EF^2$. Therefore $AE^2 = AC^2 + CE^2$ (21. II) $= \frac{2}{9}EF^2 + \frac{4}{9}EF^2 = \frac{6}{9}EF^2 = \frac{2}{3}EF^2$.

Cor. 1. *The hight of the pyramid is $\frac{2}{3}$ the diameter of the ſphere,* $EC = \frac{2}{3}EF$.

Cor. 2. *The diameter of the ſphere : diameter of the circle comprehending the baſe of pyramid : : as* $3 : \sqrt{8}$.

For $AC^2 = \frac{2}{9}EF^2$, and $4AC^2 = \frac{8}{9}EF^2$.

Cor. 3. *The area of the baſe* $ADB = EF^2 \times \frac{\sqrt{3}}{6}$.

For the area $ADB = \frac{AB^2}{4}\sqrt{3}$ (39. II). And AB^2 or $AE^2 = \frac{2}{3}EF^2$. Therefore $ADB = \frac{1}{6}EF^2\sqrt{3}$.

Cor. 4. *The radius of the inſcribed ſphere* $= \frac{1}{6}EF$.

For it is $= EC - \frac{1}{2}EF = \frac{1}{6}EF$.

PROP. XXVII.

218. *The ſolidity of a regular pyramid inſcribed in a ſphere, is* $\frac{1}{27}EF^3\sqrt{3}$.

For the ſolidity $= \frac{1}{3}EC \times$ baſe ABD (18. VI) $= \frac{2}{9}EF \times ABD$ (Cor. 1. 26) $= \frac{2}{9}EF \times \frac{1}{6}EF^2\sqrt{3}$ (Cor. 3. 26) $= \frac{\sqrt{3}}{27} \times EF^3$.

PROP.

PROP. XXVIII.

FIG. 219.

The ſquare of the diameter of a ſphere, is thrice the ſquare of the ſide of its inſcribed cube: $FA^2 = 3FD^2$.

Through the oppoſite ſides AG, DF, ſuppoſe the plane FDAG to be drawn; and through two oppoſite angles A, F, draw the diameter of the ſphere AF. Then $DA^2 = DB^2 + BA^2 = 2DB^2 = 2DF^2$ (21. II). Alſo $FA^2 = FD^2 + DA^2 = FD^2 + 2FD^2 = 3FD^2$ (ibid.).

Cor. 1. *The ſide of the cube* $DF = \frac{1}{3}FA\sqrt{3}$.

Cor. 2. *The diameter of the ſphere* AF, *is to the diameter* DA *of the circle comprehending one face of the cube; as* 1 *to* $\frac{1}{3}\sqrt{6}$.

For $FA = FD \times \sqrt{3}$, and $DA = FD\sqrt{2}$; and $FD\sqrt{3} : FD\sqrt{2} :: 1 : \sqrt{\frac{2}{3}}$, or $1 : \frac{1}{3}\sqrt{6}$.

Cor. 3. *The area of one face of the cube* DBAI *is equal to* $\frac{1}{3}FA^2$.

Cor. 4. *The ſum of the ſquares of the ſides of the inſcribed pyramid and cube, is equal to the ſquare of the diameter.*

For the former is $\frac{2}{3}$, and the latter $\frac{1}{3}$, of the ſquare of the diameter (26 and 28).

Cor. 5. *The diameter of the circle containing one face of the cube* DA, *is equal to the ſide of the pyramid.*

For $DA^2 = 2DF^2 = \frac{2}{3}FA^2$ (28) = ſquare of the ſide of the pyramid (26).

Cor. 6. *The radius of the inſcribed ſphere is* $\frac{1}{2}$ *the ſide* FD.

FIG.

PROP. XXIX.

219. *The ſolidity of a cube inſcribed in a ſphere, is* $\frac{\sqrt{3}}{9}$ *multiplied into the cube of the diameter of the ſphere:* $\frac{\sqrt{3}}{9} \times AF^3$.

For $DF = FA\sqrt{\frac{1}{3}}$, and $DF^3 = FA^3 \times \frac{1}{3}\sqrt{\frac{1}{3}} = FA^3 \times \frac{1}{9}\sqrt{3}$ (28).

Cor. *The inſcribed cube is thrice the inſcribed pyramid.*

PROP. XXX.

220. *The ſquare of the diameter of a ſphere is double to the ſquare of the ſide of an inſcribed regular octaedron* ABFDEG: $AG^2 = 2AB^2$.

Through two oppoſite angles A, G, draw the diameter AG; then the angle ABG is right (14. 4); therefore $AG^2 = AB^2 + BG^2 = 2AB^2$ (21. II).

Cor. 1. *The ſquare of the diameter of a circle comprehending a triangle of the octaedron, is* $\frac{2}{3}$ *the ſquare of the diameter of the ſphere.*

For AB^2 = thrice the ſquare of the radius (41. IV) = $\frac{3}{4}$ the ſquare of the diameter, and $AB^2 = \frac{1}{2}AG^2$ (30); therefore the diameter ſquare $= \frac{2}{3}AG^2$.

Cor. 2. *The diameter of a circle containing the triangle of the octaedron, is equal to the ſide of the pyramid.*

Cor. 3. *The ſame circle comprehends both the ſquare of the cube, and the triangle of an octaedron, inſcribed in the ſame ſphere.*

For the former diameter is $\frac{1}{3}\sqrt{6}$, and the latter $\sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$. FIG. 220.

Cor. 4. *The area of one of the faces of the octaedron, as* ABE, *is* $\frac{\sqrt{3}}{8}$ *multiplied into the ſquare of the diameter of the ſphere*; $= \frac{\sqrt{3}}{8} \times AG^2$.

For the triangle ABE $= \frac{AB^2}{4}\sqrt{3}$ (39. II) $= \frac{AG}{8}\sqrt{3}$.

Cor. 5. *The radius of the inſcribed circle is* $\frac{1}{6}$ AB $\sqrt{6}$.

For it is the perpen. from C upon ABE, ſuppoſe it = P, D = diameter of the circle encompaſſing ABE. Then $PP = \frac{1}{4}AG^2 - \frac{1}{4}DD$; and $4PP = AG^2 - D^2 = AG^2 - \frac{2}{3}AG^2$ (Cor. 1. 30) $= \frac{1}{3}AG^2 = \frac{2}{3}AB^2$, and $2P = AB\sqrt{\frac{2}{3}} = \frac{1}{3}AB\sqrt{6}$. and $P = \frac{1}{6}AB\sqrt{6}$.

PROP. XXXI.

The ſolidity of an octaedron BD, *inſcribed in a ſphere, is* $\frac{1}{6}$ *the cube of the diameter of the ſphere* AG.

For the body conſiſts of two pyramids BEDFA and BEDFG, ſtanding on the ſquare baſe BEDF, Therefore the ſolidity $= DE^2 \times \overline{\frac{1}{3}AC + \frac{1}{3}CG} = \frac{1}{2}BD^2 \times \frac{AG}{3} = \frac{1}{6}AG^3$. 220.

Cor. *A ſphere, is to the inſcribed octaedron; as the circumference of the ſphere, to its diameter.*

For the ſphere is $= \frac{2}{3}$ the circle ABGD × AG (11) $= \frac{2}{3}AG \times$ circumference ABGD $\times \frac{1}{4}AG$ (34. IV). Therefore ſphere ; octaedron : : ABGD $\times \frac{1}{6}AG^2 : \frac{1}{6}AG^3$: : ABGD : AG.

FIG.

PROP. XXXII.

221. *The ſquare of the diameter of a ſphere, is to the ſquare of the ſide of its inſcribed regular dodecaedron* DA; *as* 6 *to* $3 - \sqrt{5}$; *or as* $9 + 3\sqrt{5}$, *to* 2.

Let A be a ſolid angle of the dodecaedron; AG, AI, AL, three pentagons forming the ∠ A. Draw the diagonals, BD, BF, DF. And on the plane BDF let fall the perp. AC, and draw DC. Then $DF^2 = 3DC^2$ (41. IV), and $DC^2 = \frac{1}{3}DF^2$, and $CA^2 = DA^2 - DC^2$ (Cor. 1. 21. II) $= DA^2 - \frac{1}{3}DF^2 = DA^2 - \frac{1}{3}DA^2 \times \frac{3+\sqrt{5}}{2}$ (Cor. 3. 43. IV) $= \frac{3-\sqrt{5}}{6}DA^2$, therefore $CA = \frac{\sqrt{3-\sqrt{5}}}{\sqrt{6}}DA$. But $\frac{DA^2}{CA} =$ diameter of the ſphere (Cor. 17. IV), or the diameter $= \frac{DA^2 \times \sqrt{6}}{DA \times \sqrt{3-\sqrt{5}}} = \frac{\sqrt{6}}{\sqrt{3-\sqrt{5}}} \times DA = S$; and diameter ſquare, $SS = \frac{6DA^2}{3-\sqrt{5}}$; and $DA^2 = \frac{3-\sqrt{5}}{6}SS = \frac{2SS}{9+3\sqrt{5}}$.

Cor. 1. *The ſquare of the diameter of the ſphere, is to the ſquare of the diameter of the circle containing one face of the dodecaedron* AL; *as* 15 *to* $10 - 2\sqrt{5}$.

Let S = diameter of the ſphere, R = radius of the circle circumſcribing the pentagon, then $AD^2 = \frac{3-\sqrt{5}}{6}SS$ (32); and $RR = \frac{2AD^2}{5-\sqrt{5}}$ (44. IV) $= \frac{2}{5-\sqrt{5}} \times \frac{3-\sqrt{5}}{6}SS = \frac{1}{3}SS \times \frac{3-\sqrt{5}}{5-\sqrt{5}} = \frac{1}{3}SS \times \frac{3-\sqrt{5}}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{1}{3}SS \times$

15 +

$\frac{15+3\sqrt{5}-5\sqrt{5}-5}{25-5}=\frac{1}{3}SS\times\frac{10-2\sqrt{5}}{20}=$ FIG. 221. $\frac{5-\sqrt{5}}{30}SS$, and the ſquare of the diameter of that circle or $4RR=\frac{10-2\sqrt{5}}{15}SS$.

Cor. 2. *The area of one pentagon of the dodecaedron, is equal to* $\frac{5}{12}\sqrt{\frac{5-\sqrt{5}}{10}}$ *multiplied by the ſquare of the diameter of the ſphere.*

For let O be the center of the circle circumſcribing the pentagon AI; and OP ⊥ to FI. Then $OP^2=\frac{3+\sqrt{5}}{8}\times RR$ (Cor. 1. 44. IV) $=\frac{3+\sqrt{5}}{8}\times\frac{5-\sqrt{5}}{30}SS$; and the area FOI $=\frac{1}{2}OP\times FI=\frac{SS}{2}\sqrt{\frac{3+\sqrt{5}}{8}\times\frac{5-\sqrt{5}}{30}\times\frac{3-\sqrt{5}}{6}}$ $=\frac{SS}{2}\sqrt{\frac{4}{48}\times\frac{5-\sqrt{5}}{30}}$; and ſince there are 5 ſuch triangles in the pentagon, the pentagon $=\frac{5}{2}SS\sqrt{\frac{1}{12}\times\frac{5-\sqrt{5}}{30}}=\frac{5SS}{12}\sqrt{\frac{5-\sqrt{5}}{10}}$.

Cor. 3. *The ſide of the cube is equal to the diagonal* DF, *of the pentagon of a dodecaedron inſcribed in the ſame ſphere.*

For $DA^2=\frac{3-\sqrt{5}}{6}SS$ (32), and $DF=\frac{1+\sqrt{5}}{2}DA$ (Cor. 3. 43. IV); and $DF^2=\frac{6+2\sqrt{5}}{4}DA^2$ $=\frac{3+\sqrt{5}}{2}DA^2=\frac{3+\sqrt{5}}{2}\times\frac{3-\sqrt{5}}{6}SS=$ $\frac{9-5}{2\times 6}SS=\frac{4}{2\times 6}SS=\frac{1}{3}SS$. But the ſquare of the ſide of the inſcribed cube is alſo $=\frac{1}{3}SS$ (28).

Therefore

FIG. 321. Therefore the diagonal in the pentagon = ſide of the cube.

PROP. XXXIII.

The cube of the diameter of a ſphere, is to the ſolidity of the inſcribed regular dodecaedron; as 1, *to* $\frac{5}{6}\sqrt{\frac{3+\sqrt{5}}{30}}$.

Let S = diameter of the ſphere, R = radius of the circle encompaſſing the pentagon, P = perpendicular from the center of the ſphere upon the pentagon, then $RR = \frac{5-\sqrt{5}}{30}SS$ (Cor. 1. 32). Then $PP = \frac{1}{4}SS - RR$ (Cor. 1. 21. II) $= \frac{15-10+2\sqrt{5}}{60}SS = \frac{5+\sqrt{5}}{60}SS$, and $P = S\sqrt{\frac{5+2\sqrt{5}}{60}}$, and the area of the pentagon $= \frac{5}{12}SS\sqrt{\frac{5-\sqrt{5}}{10}}$ (Cor. 2. 32). Therefore the pyramid whoſe baſe is the pentagon, and vertex at the center of the ſphere, is $= \frac{1}{3}S^3 \times \frac{5}{12}\sqrt{\frac{5+2\sqrt{5}}{60} \times \frac{5-\sqrt{5}}{10}}$ (18. VI) $= \frac{5}{36}S^3\sqrt{\frac{25+10\sqrt{5}-5\sqrt{5}-10}{600}} = \frac{5}{36}S^3\sqrt{\frac{15+5\sqrt{5}}{600}} = \frac{5}{36}S^3\sqrt{\frac{3+\sqrt{5}}{120}}$; but as there are 12 ſuch pyramids in the body, therefore the dodecaedron $= \frac{5}{3}S^3\sqrt{\frac{3+\sqrt{5}}{120}} = \frac{5}{6}S^3\sqrt{\frac{3+\sqrt{5}}{30}}$.

Cor. *The radius of the ſphere inſcribed in the dodecaedron, is* $DA\sqrt{\frac{25+11\sqrt{5}}{40}}$; DA *being the ſide of the dodecaedron.*

For that radius is $= P = S\sqrt{\frac{5+2\sqrt{5}}{60}} = DA\sqrt{\frac{5+2\sqrt{5}}{60}} \times \frac{9+3\sqrt{5}}{2}$ (32) $= DA\sqrt{\frac{75+33\sqrt{5}}{120}} = DA\sqrt{\frac{25+11\sqrt{5}}{40}}$. FIG. 221.

PROP. XXXIV.

The ſquare of the diameter of a ſphere, is to the ſquare of the ſide of its inſcribed regular icoſihedron; as 10 *to* $5-\sqrt{5}$; *or as* $5+\sqrt{5}$ *to* 2. 222.

Let BDEFG be the pentagonal baſe of the ſolid angle A, made by 5 triangles of the icoſiedron; let AC be perp. to it, and draw DC. Then $DC^2 = \frac{2}{5-\sqrt{5}}DE^2$ (44. IV) $= \frac{2}{5-\sqrt{5}}AD^2 = AD^2 \times \frac{2}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{10+2\sqrt{5}}{25-5}AD^2 = \frac{5+\sqrt{5}}{10}AD^2$. And $AC^2 = AD^2 - DC^2$ (Cor. 1. 21. II) $= AD^2 - \frac{5+\sqrt{5}}{10}AD^2 = \frac{5-\sqrt{5}}{10}AD^2$, and $AC = AD\sqrt{\frac{5-\sqrt{5}}{10}}$. But the diameter of the ſphere $= \frac{AD^2}{AC} = \frac{AD^2}{AD\sqrt{\frac{5-\sqrt{5}}{10}}} = AD\sqrt{\frac{10}{5-\sqrt{5}}}$, and the ſquare of the diameter $= AD^2 \times \frac{10}{5-\sqrt{5}} = SS$; and $AD^2 = \frac{5-\sqrt{5}}{10}SS = \frac{2SS}{5+\sqrt{5}}$.

Cor. 1. *The diameter of the ſphere, is to the diameter of the circle comprehending five ſides of the icoſiedron; as* $\sqrt{5}$ *to* 2.

For if S = diameter of the ſphere, then $SS = AD^2 \times \frac{10}{5-\sqrt{5}}$, and $DC^2 = AD^2 \times \frac{2}{5-\sqrt{5}}$, and

FIG. 222. $4DC^2 = AD^2 \times \frac{8}{5-\sqrt{5}}$; therefore $SS : 4DC^2 :: 10 : 8 :: 5 : 4$. And $S : 2DC :: \sqrt{5} : 2$.

Cor. 2. *The ſquare of the diameter of the ſphere, is to the ſquare of the diameter of the circle containing one triangle of the icoſiedron; as* 15, *to* $10 - 2\sqrt{5}$.

For let R = radius of the circle circumſcribing the triangle ADB; then $AD^2 = 3RR$ (41. IV), and $AD^2 = \frac{5-\sqrt{5}}{10}SS$ (34); therefore $\frac{5-\sqrt{5}}{10}SS = 3RR$, and $\frac{5-\sqrt{5}}{30} \times SS = RR$, and $\frac{10-2\sqrt{5}}{15}SS = 4RR$.

Cor. 3. *The ſame circle comprehends both the pentagon of a dodecaedron, and the triangle of an icoſiedron, inſcribed in the ſame ſphere.*

Cor. 4. *The area of a triangle* ADB *of the icoſiedron, is equal to* $\frac{5\sqrt{3}-\sqrt{15}}{40}$ × *ſquare of the diameter of the ſphere.*

For the area $= \frac{DA^2}{4}\sqrt{3}$ (39. II) $= \frac{SS}{4} \times \frac{5-\sqrt{5}}{10}\sqrt{3}$ (34) $= SS \times \frac{5-\sqrt{5}}{40}\sqrt{3} = SS \times \frac{5\sqrt{3}-\sqrt{15}}{40}$.

PROP. XXXV.

222. *The cube of the diameter of a ſphere, is to the ſolidity of the inſcribed regular icoſihedron; as 6 to* $\sqrt{\frac{5+\sqrt{5}}{2}}$.

Let P = the perpendicular from the center of the ſphere, upon the triangle ADB of the icoſiedron. R = radius of the circle encompaſſing the triangle. Then $RR = \frac{5-\sqrt{5}}{30}SS$ (Cor. 1. 34).

Then $PP = \frac{1}{4}SS - RR = \frac{1}{4}SS - \frac{5-\sqrt{5}}{30}SS = \frac{5+2\sqrt{5}}{60} \times SS$, and $P = S\sqrt{\frac{5+2\sqrt{5}}{60}}$. And area of the triangle $ADB = \frac{5-\sqrt{5}}{40}\sqrt{3} \times SS$ (Cor. 4. 34). Therefore the pyramid whoſe baſe is ADB, and vertex at the center of the ſphere is $= \frac{1}{3}P \times ADB$ (18. VI) $= \frac{SS\sqrt{3}}{3} \times \frac{5-\sqrt{5}}{40} \times S\sqrt{\frac{5+2\sqrt{5}}{60}}$ (dividing by $\sqrt{3}$) $= \frac{S^3}{3 \times 40} \times \overline{5-\sqrt{5}} \times \sqrt{\frac{5+2\sqrt{5}}{20}}$ (ſquaring $5-\sqrt{5}$) $= \frac{S^3}{3\times 40} \times \sqrt{\overline{30-10\sqrt{5}} \times \frac{5+2\sqrt{5}}{20}}$ $= \frac{S^3}{3 \times 40}\sqrt{\frac{50+10\sqrt{5}}{20}} = \frac{S^3}{3 \times 40}\sqrt{\frac{5+\sqrt{5}}{2}}$. And 20 ſuch pyramids, or the icoſiedron $= \frac{S^3}{6}\sqrt{\frac{5+\sqrt{5}}{2}}$.

FIG. 222.

Cor. *The radius of the ſphere inſcribed in the icoſihedron, is* $DA\sqrt{\frac{7+3\sqrt{5}}{24}}$, DA *being the ſide of the icoſihedron.*

For that radius is $= P = S\sqrt{\frac{5+2\sqrt{5}}{60}} = DA\sqrt{\frac{5+2\sqrt{5}}{60} \times \frac{5+\sqrt{5}}{2}}$ (34) $= DA\sqrt{\frac{35+15\sqrt{5}}{120}}$ $= DA\sqrt{\frac{7+3\sqrt{5}}{24}}$.

SCHOLIUM.

A ſphere may be inſcribed or circumſcribed to any regular body, or to any triangular pyramid.

BOOK

BOOK VIII.

The conſtruction of geometrical problems.

DEFINITION.

FIG. A Problem is ſaid to be *conſtructed geometrically*, when it is done by the help only of a ſtraight ruler, and a pair of compaſſes.

PROB. I.

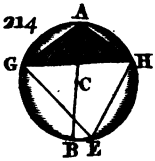
223. *To draw a ſtraight line from one point* A, *to another* B, *upon a plane.*

Set one foot of the compaſſes in the point A, and apply the edge of one end of the ruler to it; keep it cloſe there, whilſt you turn the other end of the ruler about, till the edge of it fall upon the other point B; then draw a line by the edge of the ruler, which will go from one point to the other.

PROB. II.

224. *To produce a line* AB, *that is too ſhort.*

Lay the edge of one end of the ruler againſt the foot of the compaſſes, placed at one end of the line A; and turn the other end about it, till the edge fall upon the other end of the line B. Then through B, draw a line by the edge of the ruler, from B to F.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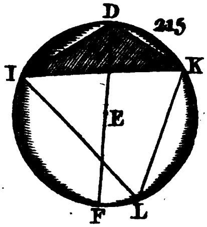
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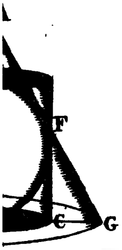
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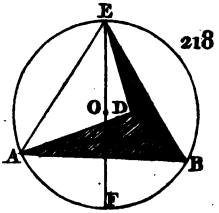
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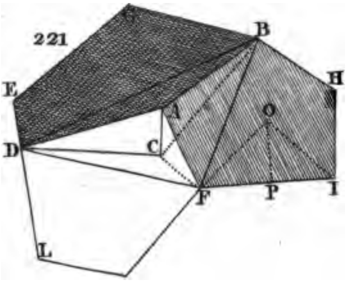
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Otherwiſe.

FIG.

Place one foot of the compaſſes in the end A, and through the other end, draw the obſcure arch CBD, with the other foot of the compaſſes, opened to the diſtance AB. In that arch take BC equal to BD; then with any opening of the compaſſes, ſeting one foot in C and D, deſcribe two obſcure arches to interſect in E; then draw BEF. 224.

For if the lines AC, AD, CE, DE, CD be ſuppoſed to be drawn; the line CD will be ⊥ to AB, and to BF (Cor. 3. 3. II); and ABF, a right line (1. I).

PROB. III.

From a given point C, *to draw a line equal to a given line* AB. 225.

Draw the line CD, ſufficiently long; then take the extent AB in your compaſſes, and ſeting one foot in C, ſtrike the obſcure arch, F. Then CF = AB.

PROB. IV.

To find the ſum and difference of two given lines AB, BD. 226.

Draw any line DA ſufficiently long, then take the ſhorter line AB in your compaſſes, and ſeting one foot in B, deſcribe two arches to cut AD in A and F; then will DA = BD + BA, and DF = BD — BA.

PROB. V.

To divide a given angle ACB *into two equal parts.* 227.

From the angular point C deſcribe any arch AB, to cut CA, CB; then with any extent, ſeting one foot in A and B, deſcribe two obſcure arches, to

cut

FIG. cut each other in D; then draw CD; and $\angle ACD$
227. $= DCB$.

For if AD, BD be ſuppoſed to be drawn; the $\angle DCA = DCB$ (8. II).

PROB. VI.

228. *To divide a given right line* AB *into two equal parts.*

From the ends A, B, with the ſame extent, deſcribe two arches, to cut one another in C, and D. Draw CD to cut AB in I. Then $AI = IB$.

For if AC, AD, BC, BD be ſuppoſed to be drawn, ACBD will be a rhombus; and $AI = IB$ (2. III).

PROB. VII.

229. *To make an angle* B, *equal to a given angle* A.

Upon the angular point A as a center deſcribe an arch FG. Draw any line BC, and from B with the ſame extent as before, deſcribe the arch CD. Make the arch $CD = FG$, and draw BD. Then $\angle CBD = \angle FAG$.

PROB. VIII.

230. *Through a given point* A, *to draw a line* AB *parallel to another* CD.

Take the neareſt diſtance of the point A from CD; and ſeting one foot in ſome point of the line CD, deſcribe an occult arch O. Then through A draw a line AB to touch the arch O; which will be $\parallel$ to CD.

Otherwiſe.

231. From ſome point O in the line CD as a center, with the diſtance OA, deſcribe a ſemicircle CABD paſſing through A; then make the arch

DB = arch CA; and draw AB, which will be ∥ to CD (Cor. 13. IV). FIG. 231.

Or thus.

With any extent, and one foot in A, deſcribe an arch to cut CD in ſome point C. And with the ſame extent, and one foot in ſome point as D, in the line CD, deſcribe an arch B to cut AB. Then with the extent CD, and one foot in A, croſs the laſt arch in B; then draw AB, which is parallel to CD (1. III). 232.

Or thus.

From a point D taken at pleaſure in the line DC, deſcribe through A, the arch AC; and from A, with the ſame extent, the arch DB. Make DB = AC. And draw AB, which will be ∥ to DC (Cor. 2. 4. I). 233.

PROB. IX.

From a given point P *in a right line* AB, *to raiſe a perpendicular.* 234.

Make PC equal to PB, and from C and B, with a convenient extent, deſcribe two arches to cut each other at D; draw DP, which will be ⊥ to CB (8. II).

Or thus.

With any diſtance PF, and one foot in P, deſcribe the circle FCD, and ſet FP from F to C, and from C to D; from the points C, D, with any extent, deſcribe two arches to interſect at O, then draw OP, which is ⊥ to AB. 235.

For FC is the third part of a ſemicircle (45. IV), and CD is biſected by OP (Cor. 3. 3. II), and alſo the arch CD (Cor. 2. 2. IV), and therefore ∠FPO = OPB = a right angle.

PROB.

FIG.

PROB. X.

236. *To raiſe a perpendicular on the end* A, *of a line given*, AB.

Set one foot in A, and extend the other to any point C out of the line AB. From C as a center deſcribe the ſemicircle PAF, to cut AB in F. Through F and C draw FCP, to cut the ſemicircle in P. Then draw PA, which will be ⊥ to AB (14. IV).

Otherwiſe.

237. From the center A, at any diſtance AF, deſcribe the arch FG; ſet AF from F to G. And from G with the ſame extent deſcribe an arch P. Through F and G, draw the line FGP, to cut the arch in P. Then draw PA, which is perpendicular to AB.

For if AG be drawn, $\angle$FAG $= \frac{2}{3}$ of a right angle (Cor. 2. 3. II) $=$ AGF $=$ 2GAP (1. II). Therefore GAP $= \frac{1}{3}$ a right angle; and the whole FAG $+$ GAP $= \frac{2}{3} + \frac{1}{3} = 1$ whole right angle.

Or thus.

238. Take any length in your compaſſes, as AC; and ſet it 5 times along the line AB, to C, E, D, I, K; take 3 parts AD in your compaſſes, and with one foot in A deſcribe an arch P; then with extent AK (or 5 parts), and one foot in I, croſs the arch P; then from the point of interſection P to A draw PA, which is ⊥ to AB (Cor. 3. 21. II).

It will be the ſame thing, if you ſet AI from A to P, and AK from D to P.

PROB. XI. FIG.

From a given point P, *to let fall a perpendicular upon a given line* AB. 239.

From the center P deſcribe an arch to cut AB in E and F. From E and F, with a proper diſtance, deſcribe two obſcure arches to interſect in I, then through P and I, draw PC; which is perp. to AB (Cor. 3. 3. II).

Or thus.

From a point A in the line AB, with diſtance AP, deſcribe the arch PI; likewiſe from another point D, in AB, with diſtance DP, deſcribe the arch PI to cut the former in I. Draw PCI, and PC is ⊥ to AB. 240.

For if AP, AI be drawn, then PC = CI, and AC ⊥ PI (Cor. 3. 3. II. and 8. II).

PROB. XII.

To divide the given line AB *into any number of equal parts.* 241.

Draw any indefinite line AP, on which ſet the equal parts AL, LM, MN, NP. Draw PB, and through L, M, N, draw LD, ME, NF ∥ to PB. Then AD = DE = EF = FB (12. II).

Otherwiſe.

From the ends A, B, of the given line, draw two lines AP, BK as long as you will, parallel to one another. Then ſet any equal parts from A towards P, and likewiſe from B towards K. Then draw lines between the correſpondent points, NG, MH, LI, which will divide AB into the equal parts AD, DE, EF, FB (12. II). 242.

FIG.

Or thus.

243. Let AB be given to be divided; draw CP ∥ to AB. Set any equal parts, from C to L, L to M, M to N, and from N to P. Draw CA and PB to interſect in G; and draw GL, GM, GN, to cut AB in D, E, F. Then AD, DE, EF, FB are all equal (Cor. 13. II).

PROB. XIII.

244. *To divide a given line* AB, *in proportion as another line* AC *is divided in* D *and* E.

Let AB and AC be joined at A, making the angle BAC; draw CB; and through D, E, draw DF, EG ∥ to CB. Then will AF : AD :: FG : DE :: GB : EC (Cor. 2. 12. II).

PROB. XIV.

245. *To find a third proportional to two given lines,* AB, AD.

Join AB, AD at A, ſo as to make an angle BAD. Produce AD, and make AC = AD, and draw CE ∥ to BD; then AE is the third proportional. For AB : AD :: AC or AD : AE (13. II).

PROB. XV.

246. *To find a fourth proportional to three given lines,* AB, AC, AD.

Let AB, AC make any angle at A, apply the third line from A to D. Draw BC, and DE ∥ to BC; then AE is the fourth proportional. For AB : AC :: AD : AE (13. II).

PROB. XVI.

FIG.

To find a mean proportional between two given lines 247.
AB, BD.

Let AD be the ſum of the two lines AB, BD (4); biſect AD in C. With center C, and radius CA or CD, deſcribe the ſemicircle AED. From B erect BE ⊥ to AD, to cut the circle in E; then BE is the mean proportional ſought.

For AB : BE : : BE : BD (17. IV).

Or thus.

Let BA be the greater, biſect it in C, and from 248.
the center C, with radius CA or CB, deſcribe the ſemicircle BEA. Let BD be the leſſer given line. Erect DE ⊥ to BA (9), to cut the circle in E, draw BE, which is a mean between BD and BA (Cor. 17. IV).

PROB. XVII.

To divide the given line AB *in extreme and mean* 249.
proportion.

Draw EAF ⊥ to AB, and make $AE = \frac{1}{2}AB$, and draw EB, and make $EF = EB$, and $AG = AF$. And G is the point of diviſion.

For $AF = EF - EA$ (Conſt.), that is, $AG = EB - EA$, and $AG + AE = EB$ (Ax. 3), that is, $AG + \frac{1}{2}AB = EB$; and $AG^2 + AG \times AB + \frac{1}{4}AB^2 = EB^2$ (10. I), and $AG^2 = EB^2 - AG \times AB - EA^2$ (becauſe $\frac{1}{4}AB^2 = EA^2$) $= AB^2 - AB \times AG$ (Cor. 21. II) $= AB \times \overline{AB - AG} = AB \times BG$, therefore AB is cut in G, in extreme and mean proportion (Def. 11. Proportion).

FIG. 249. *Cor.* $AG = AB \times \frac{\sqrt{5}-1}{2}$, *and* $BG = AB \times \frac{3-\sqrt{5}}{2}$.

For EB or EF $= \sqrt{\frac{5}{4}AB^2} = \frac{AB}{2}\sqrt{5}$, and AF or $AG = EF - \frac{1}{2}AB = AB \times \frac{\sqrt{5}-1}{2}$.

Alſo $BG = AB - AG = AB \times \frac{2-\sqrt{5}+1}{2} = AB \times \frac{3-\sqrt{5}}{2}$.

PROB. XVIII.

250. *In any triangle* ABC, *to draw a perpendicular from any angle* A *to its oppoſite ſide* CB.

About either of the other ſides AB, deſcribe a ſemicircle ADB, to cut the ſide CB in D. Draw AD, which will be ⊥ to CB (14. IV).

PROB. XIX.

251. *Upon a given line* AB, *to make an equilateral triangle.*

Take AB in your compaſſes, and with one foot in A and B, deſcribe two arches to croſs each other at C. Draw AC, BC; and ABC is the triangle.

PROB. XX.

252. *To make a triangle of three given lines* A, B, C; *of which any two muſt be greater than the third.*

Draw DE = the line A; then take B in your compaſſes, and with one foot in E deſcribe an occult arch F. Then take C in your compaſſes, and with one foot in D, croſs the former arch at F; draw DF, EF; and DEF is the triangle required.

Cor. *After the ſame manner, a triangle is made equal to a given triangle.* FIG. 252.

PROB. XXI.

To make an iſoſceles triangle ABD, *whoſe ſide is the given line* AB; *and angle at the baſe* B *or* D, *double to that at the top* A. 253.

Let AC be the greater part of the line AB divided in extreme and mean ratio (17). From the center A through B, deſcribe the circle BD; and with extent CA, and one foot in B, croſs the circle in D; and draw AD. Then ABD is the triangle ſought.

For draw CD; then ſince AB : AC :: AC : CB (Def. 11. Proportion), that is, AB : BD :: BD : BC; therefore the triangles ABD, BDC are ſimilar (14. II), and BD = DC = CA. Whence the $\angle$B or BCD = $\angle$A + CDA (1. II) = 2$\angle$A (3. II).

Cor. *The angle* A *is* $\frac{2}{5}$ *of a right angle.*

PROB. XXII.

A triangle ABC *being given; to reduce it to another of a different baſe,* AED. 254.

Let AE be the baſe propoſed, being in the ſame line AB. Draw the line CE, from the top of the given triangle, to the point E propoſed. And through $\angle$B of the given triangle, draw BD ‖ to CE; draw the line DE. Then the triangle ADE = ACB.

For triangle DBE = triangle DBC (10. II). Add ADB, then ADB + BDE or ADE = ADB + BDC or ABC.

Cor. *Thus a triangle may be reduced to another of a different hight.*

FIG.

PROB. XXIII.

255. *To divide a triangle* ABC, *in any proportion, by a line drawn from an angle* A.

Divide the baſe, or oppoſite ſide BC, in D, in the proportion given (13); to D, draw the line AD; which divides the triangle ABC, in the ſame given ratio (11. II).

PROB. XXIV.

256. *To reduce a polygon* ABCDE *to fewer ſides.*

To take away the angle B; produce the next ſide EA, then draw the diagonal CA, and from B, draw BG ∥ to CA, to cut EA in G; and draw CG, which takes in the triangle CAG, inſtead of its equal CAB (10. II). So the polygon becomes CGED.

Cor. *By thus taking away one angle after another; any polygon may, at laſt, be reduced to a triangle.*

PROB. XXV.

257. *Upon a given line* A, *to make a ſquare.*

Draw BC = A, take A or BC in your compaſſes, and with one foot in C, deſcribe the arch BED; and with one foot in B, the arch CEF. Set the ſame extent from the interſection E to F; draw CF to cut BE in G; make ED and EI = EG; and draw BI, ID, DC, and BIDC is the ſquare required.

For if CE, BE, BF be drawn, ∠BCE = $\frac{2}{3}$ a right angle (Cor. 2. 3. II) = CBE = EBF, and ∠ECF = $\frac{1}{3}$ a right one (12. IV), therefore ECD = $\frac{1}{3}$ a right angle, and BCD = a right angle.

Or

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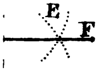

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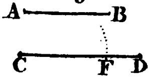

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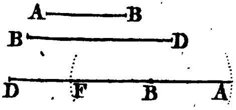

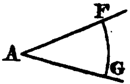

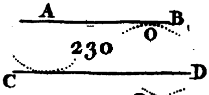

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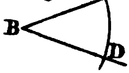

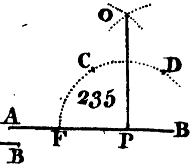

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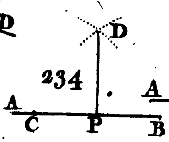

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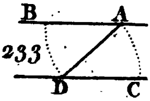

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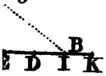

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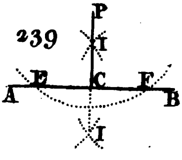

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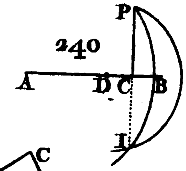

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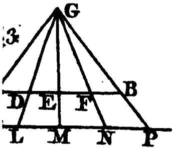

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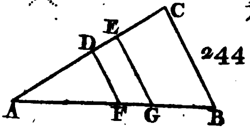

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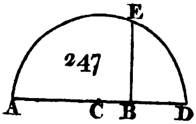

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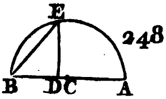

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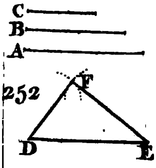

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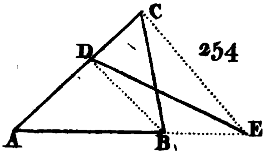

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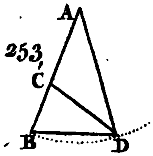

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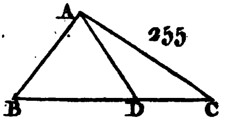

Pl.XII. *Pa.166.*

Or thus. FIG.

Make BC = A, draw CD ⊥ and = CB (9); with extent BC; and one foot in B, deſcribe an arch I; with the ſame extent and one foot in D, croſs the arch at I; draw BI, ID; then BIDC is the ſquare. 257.

PROB. XXVI.

With two given lines A, B, *to make a rectangle.* 258.

Make the baſe CD = B, draw CF perp. to CD (9), and = A; with the extent B, and one foot in F, deſcribe an arch E; and with extent A, and one foot in D, croſs the arch at E; draw FE, ED; and CFED is the rectangle ſought.

PROB. XXVII.

To make a ſquare equal to a given rectangle ABCD. 259.

Produce the ſides AD, CD, and make DF = DC; biſect AF in I, and with radius IA or IF, deſcribe the ſemicircle AEF to cut CE in E. On DE make the ſquare DH, which will be equal to the rectangle AC or AD × DF (17. IV).

PROB. XXVIII.

To make a parallelogram equal to a triangle given ABC; *and having an angle, equal to a given angle* D. 260.

Through A draw AG ∥ to BC, and make the ∠BCG = D; biſect BC in E, and draw EF ∥ to CG; then the parallelogram EG = triangle ABC (7. III).

FIG.

PROB. XXIX.

261. *Upon a given right line* A, *to make a parallelogram equal to a given triangle* B; *having an angle, equal to a given one* C.

Make a parallelogram GE = triangle B (28), whoſe angle G = C; produce DG, EF, DE, GF; and make FH = A; through H, draw IL ∥ to EF, to meet DE in I; draw IFK, to cut DG in K; through K draw KL ∥ to GH, meeting EF and IH in M and L. Then the parallelogram MH = B.

For parallelogram MH = GE (4. III) = B (Conſtr.).

Or thus.

262. Let B be the given triangle; produce the baſe, and draw FG, parallel thereto; make the ∠DCG = C, and CI = baſe of the triangle B. Then triangle CGI = B (10. II); make CD = A, and make triangle CKD = CGI (22); biſect CK in H, draw HL, DL ∥ to CD, CH; then CL is the parallelogram ſought.

For CHLD = triangle CKD (7. III) = CGI (Conſtr.) = B.

PROB. XXX.

263. *Upon a given right line* FG, *to make a parallelogram equal to a given polygon* BACD, *having an angle equal to a given one* E.

Divide the polygon into triangles CAB, CBD. Make the angle GFK = E; and make the parallelogram GI = triangle CAB, and HK = CBD (29). Then parallelogram FL = CABD.

Cor. 1. *Hence a ſquare may be made equal to any given polygon; by making a rectangle equal to the polygon, and then a ſquare equal to the rectangle* (27). FIG. 263.

Cor. 2. *Thus a parallelogram may be made equal to the ſum or difference of two given polygons.*

PROB. XXXI.

To make a ſquare equal to the ſum of two ſquares. 264.

Make FBD a right angle; make BA = ſide of one given ſquare; BC = ſide of the other ſquare, draw AC; then the ſquare made on AC, is equal to the ſum of the ſquares made upon AB, and BC (21. II).

Cor. *After the ſame manner a ſquare may be found equal to three or more ſquares. For draw* OC ⊥ *to* AC, *and equal to the ſide of a third ſquare, and draw* AO. *Then* $AO^2 = AC^2 + CO^2 = AB^2 + BC^2 + CO$ (21. II); *and ſo on.*

PROB. XXXII.

To make a ſquare equal to the difference of two ſquares. 264.

Make the right angle FBD; ſet the ſide of the leſſer ſquare from B to A; take the ſide of the greater in your compaſſes, and ſeting one foot in A, with the other croſs the line BD, in C. Then CB is the ſide of the ſquare equal to the difference of the ſquares (Cor. 21. II).

PROB. XXXIII.

To find the proportion of one polygon A *to another* B. 265.

Find two ſquares equal to the two polygons A, B (Cor. 1. 30); let CF be the ſide of the

FIG. 265. firſt, and draw FE ⊥ to it, and equal to the ſide of the ſecond. Draw the hypothenuſe CE; from F, let fall the perpendicular FD upon it: then CD : DE : : polygon A : polygon B.

For CD : DE : : CF^2 : FE^2 (Cor. 4. 20. II) : : A : B (Conſtr.).

PROB. XXXIV.

266. *To make a triangle equal and ſimilar to a given triangle* ABC.

Draw any line DE, and make DE = AB; then with extent AC, and one foot in D, deſcribe an occult arch F. And with extent BC, and one foot in E, croſs the arch at F; draw DF, EF; and DEF is the triangle required (8. II).

Or thus.

Make the ∠EDF = BAC, and DE = AB, and DF = AC, and draw EF. And DEF is the triangle ſought (6. II).

PROB. XXXV.

267. *To make a plane figure equal and ſimilar to another* ABCDEF.

In any line AF, take two marks or points M, N. Alſo in the line *af*, take *mn* = MN. With the diſtances from M to B, C, D, *&c*, and one foot in *m*, deſcribe as many arches; then with the diſtances from N to B, C, D, *&c*, and one foot in *n*, croſs them in *b*, *c*, *d*, *e*, &c. make *ma* = MA, *nf* = NF; and draw the lines *ab*, *bc*, *cd*, *de*, *ef*, in like manner as the correſpondent lines are drawn in the other figure; and it is done.

Or thus. FIG.

Let the given figure ABCDE be divided into 268.
the triangles BAC, CAD, DAE. Then make triangle GFH = BAC, HFI = CAD, and IFK = DAE (34). And the polygon GK will be equal and ſimilar to BE.

PROB. XXXVI.

To make a polygon ſimilar to another ABCDE, *and in the given ratio of* AF *to* AB. 269.

Find AG a mean proportional between AF and AB. Draw the diagonals AC, AD. Then from G, draw GH, HI, IK parallel to BC, CD, DE. And AGHIK is the polygon.

For the correſpondent triangles in both being ſimilar, the polygons are ſimilar (Cor. 2. 19. III). Alſo AF : AG : : AG : AB (Conſtr.), and AF : AB : : AG^2 : AB^2 (23. Proportion) : : polygon GI : polygon BD (20. III).

Otherwiſe thus.

Make PQ = AG; alſo make the angles QPR, 269.
RPS, SPT, reſpectively equal to BAC, CAD, DAE. 270.
And make the angles Q, R, S, T ſucceſſively = B, C, D, E. And the polygon PQRST is that ſought.

For each of the triangles in one figure is ſimilar to each in the other (Def. 10. II); and therefore the polygons are ſimilar (Cor. 1. 19. III).

Cor. *And thus a polygon is made upon a given line* AG *or* PQ, *ſimilar to another polygon* ABCDE.

FIG.

PROB. XXXVII.

270. *To make a polygon* TQ *equal to a given one* F, *and*
271. *ſimilar to another* ACDEB.
272.

Upon BA make the rectangle BM = BACDE (30); and upon BH make the rectangle BI = F (30). Take PQ a mean proportional between BA and BR (16); and upon PQ, make the polygon PQRST ſimilar to BACDE (Cor. 36); and TQ is the polygon ſought = F.

For polygon BD : polygon PS : : $BA^2 : PQ^2$ (20. III) : : BA : BR (23. Proportion) : : BM or polygon BD : BI or polygon F (8. III). Therefore polygon PS = F (Ax. 7. Proportion).

Cor. *If the polygon* TQ *was to be to* F, *in the given ratio of* R *to* S; *it is done the ſame way; only the parallelogram* BI *muſt be made* $= \frac{R}{S} \times F$.

PROB. XXXVIII.

To find the center of a circle ADF.

273. Draw any line AD, and biſect it in B; through B draw GBF ⊥ to AD. Biſect GF in C, for the center (Cor. 1. 2. IV).

Cor. *By the ſame rule, the arch* AGD *is biſected* (Cor. 1. 2. IV).

PROB. XXXIX.

274. *Through three given points* A, B, F, *to deſcribe a circle.*

Draw AB, BF, and biſect them in D, E. Thro' D and E, draw the perpendiculars DC, EC, to meet in C. C is the center (Cor. 1. 2. IV).

Cor.

FIG.

Cor. *If an arch of a circle be given; the center may be found, by taking three points in that arch. And then the circle may be compleated.* 274.

PROB. XL.

To draw a tangent to a circle from a given point A. 275.

From the point A to the center C, draw the line AC, biſect AC in D. With the radius DA or DC, deſcribe a ſemicircle to cut the circle in B. Draw AB, which will touch the circle in B (10 and 14. IV).

PROB. XLI.

Upon a right line AB, *to deſcribe the ſegment of a circle, which ſhall contain an angle* AIB, *equal to a given angle* C. 276.

Make the angle BAD = C. Through A draw AE ⊥ to AD. At the other end B, make the ∠ ABF = BAF, to cut AE in F. From the center F, with radius FA or FB, deſcribe the ſegment of a circle AIEB. Then ∠ AIB = C.

For AF = FB (Cor. 1. 3. II); and ∠ AIB = BAD (16. IV) = C.

Or thus.

Cut out a piece of wood, *&c.* with two ſtraight ſides, making an angle equal to C. And placing it between the fixt points A, B; move the angular point about, while the ſides move cloſe by the points A, B; then the angular point will deſcribe the arch AIEB.

Cor. *In the ſame manner a ſegment is cut off from a circle, to contain a given angle; by drawing the tangent* AD *at* A, *and making the angle* BAD = C. *Then* AIEB *is the ſegment.*

FIG.

PROB. XLII.

277. *In a circle* AEC, *to inſcribe a triangle ſimilar to a triangle given,* DFG.

Draw LK to touch the circle at A; make ∠KAC = F, and ∠LAB = G. Draw BC, and the triangle BAC is ſimilar to FDG (16. IV).

PROB. XLIII.

278. *In a given triangle* ABC, *to inſcribe a circle.*

Biſect two angles B, C, with the lines BD, CD, meeting in D. Let fall DF ⊥ to BC. With radius DF, and center D, deſcribe the circle FEG, which will touch all the ſides of the triangle ABC (Cor. 1. 35. II).

PROB. XLIV.

279. *About a given circle* ABC, *to deſcribe a triangle ſimilar to a triangle given,* DEF.

Produce the ſide EF both ways, to G and H. At the center I, make ∠AIB = GED, and BIC = DFH. Then to the points B, A, C, draw three tangents to the circle, to interſect in the points L, M, N. Then the triangle LMN, is ſimilar to EFD.

For ſince the ∠s at A, B, C are right angles; ∠L + AIB = 2 right angles (Cor. 16. III) = GED + DEF, and taking away the equal angles AIB, and GED; then ∠L = DEF. For the ſame reaſon M = DFE, conſequently N = D.

PROB.

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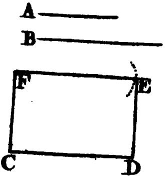

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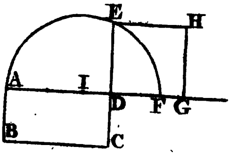

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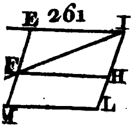

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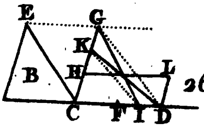

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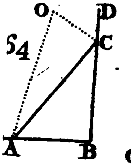

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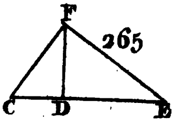

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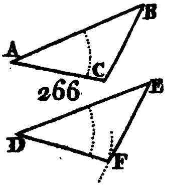

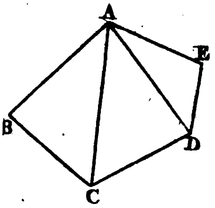

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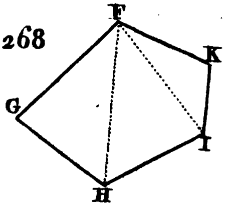

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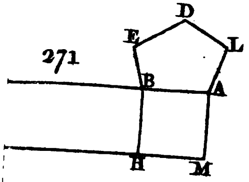

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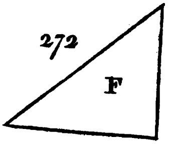

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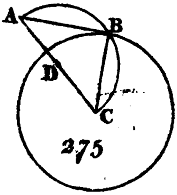

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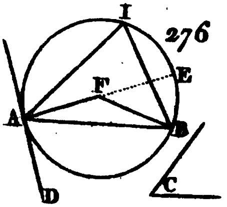

Pl. XIII. Pa. 274.

PROB. XLV.

FIG. 280.

About a triangle ABC, *to deſcribe a circle.*

Biſect any two ſides, AB, BC, in D and E. Raiſe the perpendiculars DF, EF, to interſect in F. From F as a center deſcribe a circle through B, which will paſs through A, C (Cor. 32. II).

Cor. *In an acute-angled triangle, the center is within the triangle; in an obtuſe one, without* (Cor. 1. 14. IV).

PROB. XLVI.

In a given circle FCD, *to inſcribe an equilateral triangle.* 281.

Draw the diameter FB. With the radius BA and center B, deſcribe two arches C, D, to cut the circle in C and D. Draw the lines CD, DF, FC. And CFD is an equilateral triangle.

For arch CB or BD $= \frac{1}{6}$ the circumference (45. IV); therefore CBD $= \frac{1}{3} =$ CF $=$ FD.

PROB. XLVII.

In a given circle ABCD, *to inſcribe a ſquare, or regular octagon.* 282.

Draw the diameters AC, BD at right angles to one another, cuting the circle in A, B, C, D. Draw the lines AB, BC, CD, DA; and ABCD is a ſquare (Cor. 2. 6. IV).

If the diameters FG, HI be drawn, biſecting the arches AFB, AHD, DGC, CIB. Then AF, or FB, &c. will be the ſide of the octagon.

Cor. *If* AF, FB, *&c. be biſected, a polygon of* 16 *ſides, will be inſcribed; and ſo on.*

FIG.

PROB. XLVIII.

283. *In a given circle* ADBG, *to inſcribe a regular pentagon, or decagon.*

Draw the diameter AB; from the center C draw CD ⊥ to AB; biſect CB in E; and make EF = ED, and draw DF, which will be the ſide of the pentagon; therefore if DH, HG, &c. be made = DF, DHGIK will be the pentagon required. Alſo FC is the ſide of the decagon; therefore if DL, LK, &c. be made = CF; a regular decagon will be inſcribed.

For $DF^2 = DE^2 + EF^2 - 2FE \times EC$ (23. II) $= 2DE^2 - 2DE \times EC = 2DE^2 - DE \times DC$. But $DE^2 = DC^2 + CE^2$ (21. II) $= \frac{5}{4}DC^2$, and $DE = \frac{1}{2}DC\sqrt{5}$. Therefore $DF^2 = \frac{5}{2}DC^2 - \frac{DC^2}{2}\sqrt{5} = DC^2 \times \frac{5-\sqrt{5}}{2}$. Therefore DF is the ſide of a pentagon (44. IV). And FC is the ſide of a decagon (48. IV).

PROB. XLIX.

284. *In a given circle* ACE, *to inſcribe a regular hexagon.*

Make AB, BC, CD, DE, EF and FA, all equal to the radius AG: and drawing the lines, the figure ABCDEF is a hexagon (45. IV).

Cor. *If the arch* AB *be biſected, you will have the ſide of a regular dodecagon.*

PROB.

PROB. L.

FIG. 285.

About a given circle ABC, *to deſcribe a regular polygon.*

Either inſcribe a polygon of the ſame ſort, or divide the circle into ſo many equal parts AB, BC, *&c.* as the polygon has ſides. To the points of diviſion, draw the radii GA, GB, GC, *&c.* To theſe lines at A, B, C, *&c.* draw tangents to the circle, KD, DE, EF, *&c.* to interſect in D, E, F, *&c.* then DEFHIK is the polygon required.

For if GD, GK be ſuppoſed to be drawn, AK = AD, and ∠K = ∠D (7. II). Alſo DA = DB (Cor. 4. 21. IV), whence KD = DE = EF, *&c.*

PROB. LI.

To inſcribe a circle in any regular polygon; or deſcribe a circle about it. 285.

Biſect any two adjoining angles D, K, with the lines DG, KG, and they will meet in the center G. Or biſect any two adjoining ſides, DK, DE, with the perpendiculars AG, BG, which will meet in the center G. Take GA the neareſt diſtance to any ſide, and from G deſcribe the circle ABC, which will touch all the ſides of the polygon DH.

Likewiſe biſect any two angles A, B, with the lines AG, BG, which will meet in the center G. Or biſect any two ſides CD, DE, with two perpendiculars meeting in G, the center. Then from A with diſtance GA deſcribe a circle ABCE, which will paſs through all the angles of the figure. 284.

Cor. *A circle may be inſcribed, or circumſcribed, to any regular polygon.*

FIG.

PROB. LII.

286. *To deſcribe a polygon in one circle* ABDE, *which*
287. *ſhall be ſimilar to a polygon* FGI, *deſcribed in another,* GIK; *regular or irregular.*

Draw lines from the center P, to all the angles of the polygon, as PF, PK, PI, *&c.* Then at the center O, of the other circle, make the angles AOE, EOD, DOC, COB, BOA, reſpectively equal to FPK, KPI, IPH, HPG, GPF, Draw lines between the points A, E, D, *&c.* Then ABCDE is ſimilar to FGHIK (Cor. 1. 19. III).

Cor. *After the ſame manner, a polygon may be deſcribed about one circle, ſimilar to a polygon deſcribed about another circle.*

PROB. LIII.

288. *From a given point* A *on high; to let fall a perpendicular to a plane* BC.

In the plane BC draw any line DE. From A draw AF ⊥ to DE. Through F, draw FH ⊥ alſo to DE. Then let fall AI perp. to FH. Then AI is ⊥ to the plane BC.

For DE is ⊥ to the plane AFI (4. V). And if KL be ∥ to DE, then KL is ⊥ to the plane AFI (6. V). Therefore AI is ⊥ to the plane HIL or BC (4. V).

Otherwiſe thus.

289. Deſcribe a circle BFD, from the point A, upon the plane, with a pair of compaſſes or a ſtring. Then find the center C of that circle (37, 38. VIII); and AC is ⊥ to the plane. In practice you need only extend from A, to three points of the plane, B, D, F.

PROB.

PROB. LIV.

FIG.

From a given point A, *in a plane* BC, *to raiſe a perpendicular.* 290.

From ſome point D, above the plane, draw DE ⊥ to the plane (52). Draw AE, and draw AF ∥ to ED. Then AF is perp. to the plane BC (6. V).

Both this and the laſt Prob. may eaſily be done with two ſquares: ſetting them croſs one another, and both of them cloſe to the point A.

PROB. LV.

To draw one plane parallel to another DE, *at a given diſtance.* 291.

Take three points A, B, C in the plane DE, but not in a right line. At theſe points erect three perpendiculars AI, BK, CL, to the plane DE (53); and of equal lengths, the ſame as the given diſtance. Through I, K, L, draw the plane FG, and it will be parallel to DE.

PROB. LVI.

To draw a plane perpendicular to a right line AB, *at* B. 292.

Draw two lines CD, EF perp. to AB at B. Through C, E, D, draw the plane CEDF, which is ⊥ to AB (4. V).

PROB. LVII.

FIG. 293. *Through any two lines* AB, CD, *inclined to one another, which do not interſect; to draw two planes perpendicular to one another.*

Through any point E of the line AB; draw EF ∥ to CD. Through the lines AEF, let the plane AEBF be drawn. From any point C, in the line CD, let fall the perp. CI, upon the plane AFB. Draw IH ∥ to FE, to interſect AB in H. At H let fall HG ⊥ to CD. Then the plane CIHG will be perp. to the plane AFH.

For CD is ∥ to IH (8. V). And ſince CI is perp. to IH, it is alſo ⊥ to CG (3. I). Therefore CI, HG are parallels (Cor. 3. 4. I); and HG ⊥ to the plane AFB (6. V). Therefore the plane DCIH is perp. to the plane AFB (7. Def. V).

Cor. 1. *The right line* GH *is perpendicular to both lines* AB, CD.

For it is ⊥ to CD (Conſtr.), and it is ⊥ to the plane AHI, and therefore to AHB.

Cor. 2. GH *is the neareſt diſtance between the two lines* AB *and* CD.

For the point H is nearer G, than any other point in the line AB (Cor. 4. 21. II). And G is nearer H than any point in CD.

Cor. 3. *Hence no two lines can poſſibly be drawn; but another line may be drawn, which is perp. to them both.*

Cor. 4. *And no two lines can be drawn, but two planes may be drawn through them, perpendicular to one another.*

Cor. 5. *The given line* CD, *is parallel to the plane* AFB, *paſſing through the other line* AB. FIG. 293.

For it is parallel to HI.

PROB. LVIII.

Through any two inclined lines, which cut not one another, AB, CD; *to draw two parallel planes through them.* 293.

Draw the plane HICD and BIFA perp. to one another, and paſſing through the two given lines AB, CD (56). Then through CG at the diſtance GH, draw a plane ⊥ to GH (54), and it will be parallel to the plane ABF (Def. 10. V).

Cor. 1. *The line* GH, *(which is perpendicular to both the given lines,* AB, CD*), is the diſtance of the two parallel planes.*

Cor. 2. *No two lines can be drawn, but there may be two planes drawn through them, parallel to one another.*

PROB. LIX.

To make a ſolid angle BAD, *of three given plane angles, whoſe ſum is leſs than four right angles, and any two greater than the third.* 294.

There is no more to do than to join all their ſides AB, AC, AD, together; ſo that the vertices or angular points may all meet together in A; then A is the ſolid angle required (Cor. 19. V).

PROB.

FIG.

PROB. LX.

295. *To make a ſolid angle, equal to any ſolid angle given,* A.

Cut off the given ſolid angle A, by a plane BCDE; and from the given planes, make the angles QPR, RPS, SPT, and TPQ reſpectively equal to BAC, CAD, DAE, and EAB; alſo make PQ, PR, PS, PT reſpectively equal to AB, AC, AD, AE. Then the plane triangles in one, will be equal to the triangles in the other. Then place the ſides PR, PS, &c. together as in the other ſolid angle A, ſo that all their angular points may meet in P; and likewiſe ſo that the angles Q, R, S, T, may be reſpectively equal to B, C, D, E. And then the ſolid angle P will be equal to the ſolid angle A.

For all the 3 angles at Q, being equal to thoſe at B; and all the three angles at R, equal to thoſe at C, &c. The ſolid angles at B, C, D, E, will be equal to thoſe at Q, R, S, T (Cor. 19. V). And conſequently ∠P muſt be equal to A.

PROB. LXI.

296. *Upon a given line* AB, *to deſcribe a parallelopipedon, ſimilar to a given parallelopipedon* CD.

Make the ſolid angle A equal to the ſolid angle C (59); alſo make as CF : CE : : AB : AH; and CF : CG : : AB : AI. Then finiſh the parallelopipedon AK, by drawing the planes KI, KH, and KB, parallel to the oppoſite ones BH, BI, and IH. Then IB is ſimilar to GF. FIG. 297.

For their ſolid angles, are equal, and the ſides proportional, and therefore they are ſimilar (22. VI).

FINIS.

ERRATA.

Page	*Line*	*Read*
107	16	DFKHCILG =
121	2	Fig. 191.
	25	DCH, be 3 —— Fig. 192.
125	18	Fig. 195.
126	2, 8 19, 29	Fig. 195, 196, 197, 198.
127	2	Fig. 198.
128	2	Fig. 198.
	28, 9	Fig. 199, 200.
129	1, 2	Fig. 199, 200.
137	5	Fig. 209.

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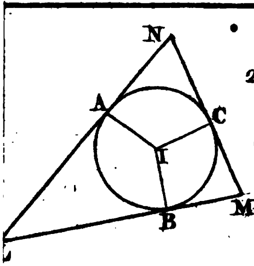

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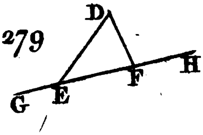

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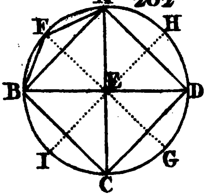

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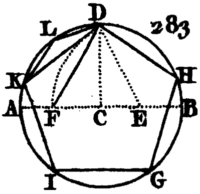

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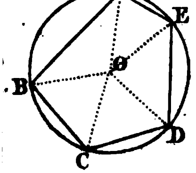

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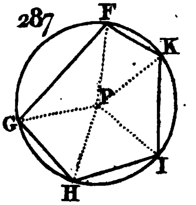

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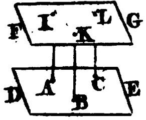

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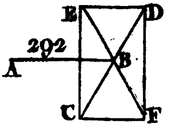

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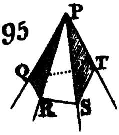

Pl.XIV. *Pa. last*